Gaussian Elimination

Thursday 13 March 2025 21:21

Defn [Algorithm]

A step by-step set of instructions to solve a problem or complete a task.

These steps must be
- (a) mechanical, precise
- (b) Works everytime
- (c) must end

* People often use it in the content of Computers ... mechanical, finite

* However algorithms are independent of Computers.

Example 1 [Make a sandwich]

Step 1 Take 2 slices of bread

Step 2 Put whatever you want on both slices

Step 3 Put slices together

mechanical + finite

Step 4 END

Other examples - Complicated recipes, putting on a tie, etc.

Example 2 [Compute Square root of X]

Step 1 Start with guess x_1

Step 2 $x_{n+1} = \frac{1}{2}\left(x_n + \frac{X}{x_n}\right)$

Step 3 repeat step 2 until values don't change a lot.

Babylonian Square Root Algorithm

$X = 7$

$x_1 = 2$

$x_2 = \frac{1}{2}\left(2 + \frac{7}{2}\right) = \frac{1}{2}\left(2 + 3.5\right) = 2.75$

$x_3 = \frac{1}{2}\left(2.75 + \frac{7}{2.75}\right) \simeq 2.65$

$x_4 = \frac{1}{2}\left(2.65 + \frac{7}{2.65}\right) \simeq 2.65$

$\boxed{2.65 \times 2.65 = 7.0225 \checkmark}$

Exercise Find Square root of 16

by starting at:

$x_1 = 3, 5, 4$

$x_1 = 3$

$x_2 = \frac{1}{2}\left(3 + \frac{16}{3}\right) \simeq 4.17$

$x_3 = \frac{1}{2}\left(4.17 + \frac{16}{4.17}\right) \simeq 4.003$

$x_4 = \frac{1}{2}\left(4.003 + \frac{16}{4.003}\right) \simeq 4.0000001$

$x_1 = 5$

4.1

4.001

4.00000…1 ✓

$x_1 = 4$

4

4

4

⋮

Homework Write an algorithm to test if a number is prime

Above algorithm:
Easy to follow, works every time, ends

NO NEED* OF COMPUTERS

Completely Mathematical

→ This course is about Algorithms in algebra

→ Plan to Cover [A] Gaussian Elimination
[B] Grobner Basis
[C] Identity testing

→ Some algorithms are better that other algorithms for the same problem

Better ⟹ Less no. of steps,
Less space, etc.

Gaussian Elimination

Algorithm to solve a system of linear equations.

Def [Matrix] Rectangular array of numbers

e.g. $\begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 9 \end{bmatrix}$, $\begin{bmatrix} 1 & -5 \\ 2 & -2 \end{bmatrix}$,

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 2 & 2 \end{bmatrix}$$

[Identity matrix]

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

[Triangular form]

$$\begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \quad \begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix}$$

$$\begin{bmatrix} x & x & x & x & \cdots \\ 0 & x & x & x & \\ 0 & 0 & x & x & \vdots \\ 0 & 0 & 0 & x & \end{bmatrix}$$

e.g. $\begin{cases} x + y = 5 \\ 2x + 3y = 11 \end{cases}$

(Step-1) Write down augmented matrix

e.g. $\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 2 & 3 & 11 \end{array}\right]$

(Step-2) Perform row operations: to transform Augmented matrix to triangular form, and then identity

[A] Swapping rows

[B] Multiply a row by a non zero number

[C] Adding a scalar multiple of one row to another

e.g. $R_1 \leftarrow 2R_1$

(1)
$$\left[\begin{array}{cc|c} 2 & 2 & 10 \\ 2 & 3 & 11 \end{array}\right]$$

(2) $R_2 \leftarrow R_2 - R_1$

$$\left[\begin{array}{cc|c} 2 & 2 & 10 \\ 0 & 1 & 1 \end{array}\right]$$

row echelon

(3) $R_1 \leftarrow R_1/2$

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 1 \end{array}\right]$$

(4) $R_1 \leftarrow R_1 - R_2$

$$\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 1 \end{array}\right]$$

Identity matrix

(Step-3) Read off solutions

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$ ✓

Example

$$2x + y - z = 8$$

$$-3x - y + 2z = -11$$

$$-2x + y + 2z = -3$$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ -3 & -1 & 2 & -11 \\ -2 & 1 & 2 & -3 \end{array}\right]$$

$$R_2 \leftarrow R_2 + \frac{3}{2} R_1$$

$$R_3 \leftarrow R_3 + R_2$$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ 0 & 1/2 & 1/2 & 1 \\ 0 & 2 & 1 & 5 \end{array}\right]$$

I can try $R_3 \leftarrow R_3 - 2R_1$, but trouble.

$$R_3 \leftarrow R_3 - 4R_2$$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ 0 & 1/2 & 1/2 & 1 \\ 0 & 0 & -1 & 1 \end{array}\right]$$

$$R_2 \leftarrow 2R_2 + R_3$$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & -1 & 1 \end{array}\right]$$

$$R_1 \leftarrow R_1 - R_2$$

$$\left[\begin{array}{ccc|c} 2 & 0 & -1 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & -1 & 1 \end{array}\right]$$

$$R_1 \leftarrow R_1 - R_3$$

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & -1 & 1 \end{array}\right]$$

$$R_1 \leftarrow R_1/2$$

$R_3 \leftarrow -R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$ ✓

Exercise

Balance the equation

$$H_2SO_4 + Al(OH)_3 \longrightarrow Al_2(SO_4)_3 + H_2O$$

$$a\,H_2SO_4 + b\,Al(OH)_3 \longrightarrow c\,Al_2(SO_4)_3 + d\,H_2O$$

Aluminium:

$$b = 2c$$

Sulphur:

$$a = 3c$$

Oxygen:

$$4a + 3b = 12c + d$$

Hydrogen

$$2a + 3b = 2d$$

So we have to solve the system

$$b - 2c = 0$$
$$a - 3c = 0$$
$$4a + 3b - 12c - d = 0$$
$$2a + 3b - 2d = 0$$

$\boxed{\text{Set } a = 1}$ ← with chemical eqns., we can freely set one variable

~~$b - 2c = 0$~~

$3c = 1$

$-3b + 12c + d = 4$

~~$3b - 2d = -2$~~

$$6c - 2d = -2$$

$$\left[\begin{array}{ccc|c} 0 & 3 & 0 & 1 \\ -3 & 12 & 1 & 4 \\ 0 & 6 & -2 & -2 \end{array}\right]$$

$R_1 \longleftrightarrow R_2$

$$\left[\begin{array}{ccc|c} -3 & 12 & 1 & 4 \\ 0 & 3 & 0 & 1 \\ 0 & 6 & -2 & -2 \end{array}\right]$$

$R_3 \leftarrow R_3 - 2R_2$

$$\left[\begin{array}{ccc|c} -3 & 12 & 1 & 4 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & -2 & -4 \end{array}\right]$$

$R_1 \longleftarrow 2R_1 + R_3$

$$\left[\begin{array}{cccc} -6 & 24 & 0 & 4 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & -2 & -4 \end{array}\right]$$

$R_1 \leftarrow R_1 - 8R_2$

$$\left[\begin{array}{ccc|c} -6 & 0 & 0 & -4 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & -2 & -4 \end{array}\right]$$

$R_1 \leftarrow -\frac{R_1}{6}$; $R_2 \leftarrow \frac{R_2}{3}$, $R_3 \leftarrow -\frac{R_3}{2}$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$a=1$, $b=2/3$, $c=1/3$, $d=2$

Mult. by 3

$a=3$, $b=2$, $c=1$, $d=6$

$$\boxed{3\,H_2SO_4 + 2\,Al(OH)_3 \rightarrow Al_2(SO_4)_3 + 6\,H_2O}$$

Homework

$$\left.\begin{array}{l} x + 2y + 3z + w = 10 \\ 2x + 3y + 4z + 2w = 20 \\ 3x + 4y + 5z + 3w = 30 \\ 4x + 5y + 6z + 4w = 40 \end{array}\right\}$$

Determinant

Convention

$$\begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} & \cdots \\ x_{2,1} & x_{2,2} & x_{2,3} & \cdots \\ x_{3,1} & & & \\ \vdots & & & \\ \vdots & & & \end{bmatrix}$$

Def [$\overset{2\times 2}{\text{Determinant}}$] defined for

— square matrices. "Recursive defn"

for 2×2

$$A = \begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{bmatrix}$$

$$\det A := x_{1,1}x_{2,2} - x_{1,2}x_{2,1}$$

e.g $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$

$$\det = (2\times 4) - (-1\times 3)$$

$$= 11$$

Exercise

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 9 & 18 \end{bmatrix}$$

Def [Determinant 3×3]

$$A = \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,2} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{bmatrix}$$

$$\det(A) := (-1)^{1+1} \cdot x_{1,1} \det \begin{pmatrix} x_{2,2} & x_{2,3} \\ x_{3,2} & x_{3,3} \end{pmatrix}$$

$$+ (-1)^{1+2} \cdot x_{1,2} \det \begin{pmatrix} x_{2,1} & x_{2,3} \\ x_{3,1} & x_{3,3} \end{pmatrix}$$

$$+ (-1)^{1+3} \cdot x_{1,3} \det \begin{pmatrix} x_{2,1} & x_{2,2} \\ x_{3,1} & x_{3,2} \end{pmatrix}$$

e.g. $\begin{pmatrix} 1 & -1 & 0 \\ 2 & 4 & 0 \\ 9 & 1 & 5 \end{pmatrix}$

$$\det = 1. \det \begin{pmatrix} 4 & 0 \\ 1 & 5 \end{pmatrix} + (-1)(-1) \det \begin{pmatrix} 2 & 0 \\ 9 & 5 \end{pmatrix}$$

$$+ 0$$

$$= 1 \cdot 20 + 10 = 30$$ ✓

Exercise
$$\begin{bmatrix} 1 & 3 & -7 \\ 1 & 4 & 9 \\ 3 & 12 & 1 \end{bmatrix}$$

defn [det 4×4]

$$A = \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} \\ x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} \\ x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} \end{bmatrix}$$

define $A_{i,j}$ to be A with i^{th} row removed and j^{th} column.

$$\det(A) = (-1)^{1+1} \cdot x_{1,1} \cdot \det(A_{1,1})$$

$$+ (-1)^{1+2} x_{1,2} \cdot \det(A_{1,2})$$

$$+ (-1)^{1+3} x_{1,3} \cdot \det(A_{1,3})$$

$$+ (-1)^{1+4} x_{1,4} \cdot \det(A_{1,4})$$

$$\vdots$$

This generalizes.

$$\det\begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & & \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix}$$

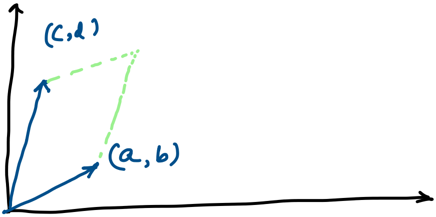
$$:= \text{Sum of } (-1)^{1+i} x_{1,i} A_{1,i}$$

for $i = 1, \ldots, n$

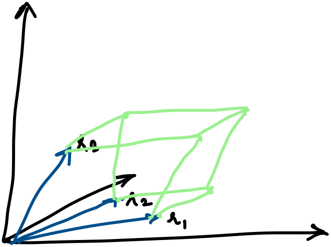
Geometry

$\det\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is the area of

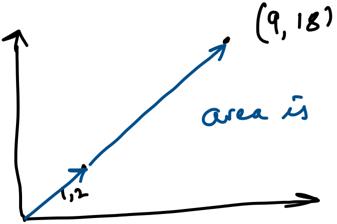
Parallelogram



$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$



recall $\det\begin{pmatrix} 1 & 2 \\ 9 & 18 \end{pmatrix} = 0$



area is indeed 0.

homework

$$\det\begin{pmatrix} 1 & -1 & 3 & 4 \\ 1 & 1 & 4 & 2 \\ 2 & -9 & 5 & 1 \\ 2 & -3 & -3 & 13 \end{pmatrix} = ?$$

In general, doing determinant this way takes around $n!$ steps for $n \times n$ matrices. HUGE

even for 60×60 matrices, impossible!

$60! \sim 10^{81} \gg$ no. of particles in the observable universe

Having said that, 60×60 is very small!!

Question Is there a better way to compute the determinant?

Answer Gaussian Elimination

A Gaussian Elimination type Algorithm to compute the determinant

Step 1 Reduce matrix to triangular form ensuring, for any

$R_i \leftarrow aR_i + bR_j$, a, b scalars,

$\boxed{a = 1}$

Step 2 Determinant is the product of the elements of the diagonal

e.g. $\begin{bmatrix} 1 & 3 \\ 2 & 9 \end{bmatrix}$

$R_2 \leftarrow R_2 - 2R_1$

$\begin{bmatrix} \boxed{1} & 3 \\ 0 & \boxed{3} \end{bmatrix}$

$\det = 1 \times 3 = 3$

Check $\det = 1 \times 9 - 3 \times 2 = 3$

e.g. $\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$

$R_2 \leftarrow 2R_2 - R_1$ ✗ not allowed

$R_2 \leftarrow R_2 - \frac{1}{2} R_1$

$$\begin{bmatrix} 2 & 3 \\ 0 & 7/2 \end{bmatrix}$$

$\det = 2 \times 7/2 = 7$

Check : $2 \times 5 - 3 \times 1 = 7$ ✓

__exercise__ $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$

example $\begin{bmatrix} 2 & 3 & -4 \\ 2 & 1 & -12 \\ 3 & 5 & 9 \end{bmatrix}$

$R_2 \leftarrow R_2 - R_1$

$$\begin{bmatrix} 2 & 3 & -4 \\ 0 & -2 & -8 \\ 3 & 5 & 9 \end{bmatrix}$$

$R_3 \leftarrow R_3 - \frac{3}{2} R_1$

$$\begin{bmatrix} 2 & 3 & -4 \\ 0 & -2 & -8 \\ 0 & 1/2 & 15 \end{bmatrix}$$

$$R_3 \leftarrow R_3 + \frac{R_2}{4}$$

$$\begin{bmatrix} 2 & 3 & -4 \\ 0 & -2 & -8 \\ 0 & 0 & 13 \end{bmatrix}$$

$$\det = -52$$

homework Check if -52 is correct

☆ This algorithm, for $n \times n$ matrix, takes

$\sim n^3$ steps

n^3 vs $n!$

$\uparrow$

Can easily handle $n = 10000$ even.

homework compute det of

$$\begin{bmatrix} 1 & 3 & 2 & 1 & 9 \\ -2 & 7 & 17 & -4 & 8 \\ 3 & 1 & 2 & 1 & 12 \\ 1 & 0 & 9 & 12 & -3 \\ 8 & 5 & 2 & -3 & 6 \end{bmatrix}$$

using Gaussian elim. and Check online to see if your answer is Correct!

* * *

$$\det \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} = x_{1,1} x_{2,2} - x_{1,2} x_{2,1}$$

$$\det \overset{A}{\begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & & \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix}}$$

$$= \underset{i=1 \text{ to } n}{\text{Sum over}} \quad (-1)^{1+i} x_{1,i} \det(A_{1,i}),$$

where $A_{1,i}$ is the matrix with 1st row and ith column removed

★ Naively calculating the determinant required $\sim n!$ steps, but a clever algorithm helped compute det in $\sim n^3$ steps.

defn [Permanent of a matrix]

$$\text{Perm}\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} := x_{1,1}x_{2,2} + x_{2,1}x_{1,2}$$

$$A := \begin{bmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & & \\ x_{n,1} & \cdots & x_{n,n} \end{bmatrix}$$

$$\text{Perm}(A) = \underset{i=1 \text{ to } n}{\text{Sum over}} \; x_{1,i}\, \text{Perm}(A_{1,i})$$

★ LITERALLY, THE ONLY DIFFERENCE between det and per IS THE ABSENCE OF $(-1)^{1+i}$ ★

	naive comp.	Clever algo.
determinant	$n!$	n^3
Permanent	$n!$	NO ONE KNOWS ANYTHING MUCH BETTER THAN $n!$???

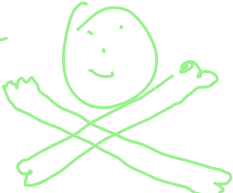
If you can find something better for permanent computation, OR, you prove nothing better is possible, then

① You'll have resolved one of the most intriguing mysteries in maths today

② You'll be resolving a huge number of other problems

③ You've solved the algebraic version of P vs NP, a clay millenium prob., that will win you 1 mil. USD

④ You'll be one of the most known mathematicians of this Century.

DO NOT TRY THIS AT HOME