Lec 1 (leftover)

Wednesday, 10 May 2023 03:02

Plan: finish overview of topics

Geometric Complexity Theory (GCT)

"String theory of Computer Science"

$V - (\mathbb{C}^{m^2})^*$, $GL(V)$ - group of automorphisms acts on

$Sym^m(V)$ - degree m homogeneous polynomials in m^2 vars.

$$\underset{GL(V)}{\underset{\cap}{L}} \cdot \overset{\in Sym^m(V)}{P(x)} = P(L^T x) \qquad \left(L_1 \cdot (L_2 P) = (L_1 L_2) \cdot P \right)$$

Observe $det_m \in Sym^m(V)$

Let $n \leq m$. $Per_n \notin Sym^m(V)$. Define padded permanent

$$Per^*_{m,n} = x^{m-n}_{m,m} \, Per_n \quad \in Sym^m(V)$$

<u>Conjecture</u> [MS] If $m = 2^{n^{O(1)}}$, then for $\forall n \geq n_0$ → sufficiently large

$$Per^*_{m,n} \notin \overline{GL(V).det_m}$$

<u>Thm</u> [MS] $dc(perm_n) \leq m \implies Per^*_{m,n} \in \overline{GL(V).det_m}$

Need orbit closures bcoz $GL(V).det_m$ contains irred polynomials

<u>Thm</u> (a) if P, Q are permutation matrices s.t $Per(A)\,Per(B) = 1$, and A, B are diagonal matrices, then

$$f(x) = f(PXQ) = f(AXB) \implies f(x) \text{ is a } \mathbb{C}\text{-multiple of the Permanent}$$

(b) if A, B are such that $det(A)\,det(B) = 1$, then

$$f(x) = f(AXB) \implies f(x) \text{ is a } \mathbb{C}\text{-multiple of the determinant}$$

$$R_{det} = \mathbb{C}\left[\overline{GL(V).det_m}\right], \quad R_{per} := \mathbb{C}\left[\overline{GL(V).Perm^*_{m,n}}\right]$$

$GL(V)$ acts on both rings

$$A \cdot q(p(x)) := q(p(A^T x))$$

Thus, by the above action, we get two representations of $GL(V)$

$$\rho_{det} \quad \& \quad \rho_{per}$$

Let $\lambda_{per}(\rho)$ denote the multiplicity of the irred representation ρ in the isotypic decomposition of ρ_{per}. Similarly $\lambda_{det}(\rho)$ ...

Thm Suppose there exists an irrep ρ s.t $\lambda_{per}(\rho) > \lambda_{det}(\rho)$. Then

$$Per^*_{m,n} \notin \overline{GL(V).\,det_m}$$

Hope: use algorithms to calculate multiplicities of irreps.

Initial Conjecture. $\exists \rho$ irrep $\lambda_{det}(\rho) = 0$ $\lambda_{per}(\rho)$ very large

"no-occurrence obstruction" X

ULRICH COMPLEXITY

Defn $uc(f)$ is the smallest r s.t there exists a matrix M of linear forms with ① $\det M = f^r$ and

② $\exists N$ s.t $M.N = f I$

Thm $VP \neq VNP \implies uc(perm_n) \geq 2^{n-2}$

$$dc\left(\sum_{i=1}^{c} x_i y_i\right) \leq c+1, \quad uc\left(\sum_{i=1}^{c} x_i y_i\right) = 2^{\lceil c/2 \rceil - 2}$$

== Connection to Ulrich Sheaves/Modules ==

$f \in k[x_0, \ldots x_n]$ homogeneous. Standard grading $\deg x_i = 1$. Let S be the graded ring. $R := S/\langle f \rangle$. Let F be a finitely generated the graded ring. $R := S/\langle f \rangle$. Let F be a finitely generated R-module. F is an Ulrich module if F has a free resolution of the form

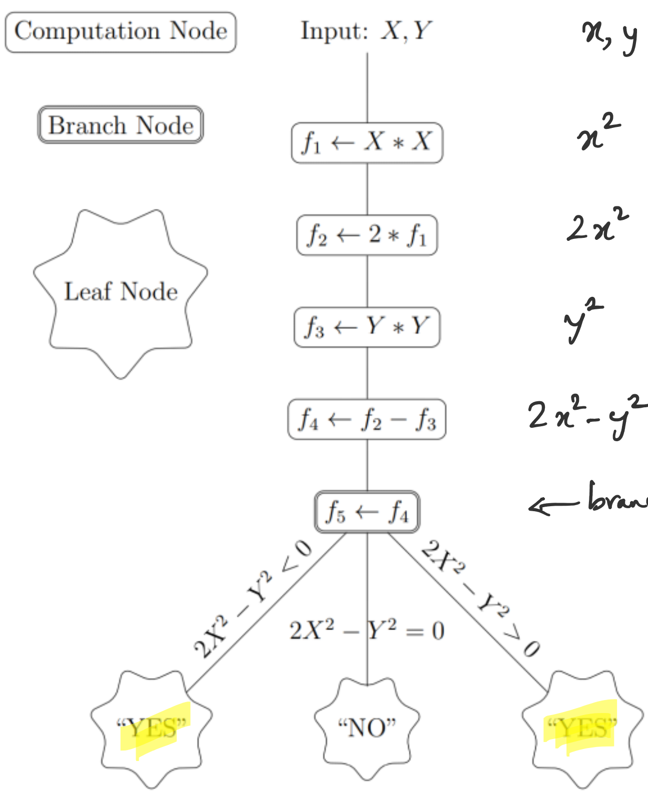
$$0 \to S^n(-1) \xrightarrow{M} S^n \to F \to 0$$

M - matrix of linear forms. Then, turns out

$$uc(f) = \inf\{ \operatorname{rank} F \mid F \text{ is Ulrich module on } R\}$$

MISC TOPICS

— Algebraic Computation tree — Model of computation that represents the computational steps that a Turing machine would execute



$x, y \in \mathbb{R}$

x^2

$2x^2$

y^2

$2x^2 - y^2$

← branch

tests for membership in the semi-algebraic set $2x^2 - y^2 \neq 0$

$x^3 + 3x^2 + 3x + 1 = (x+1)^3$

Q: what is the best way to compute? $\Longleftrightarrow$ what is the least height of ACT for the problem?

Thm [Gabrielov-Vorobjov] Consider the problem of testing membership in a semi-algebraic set $S \subseteq \mathbb{R}^n$. $\exists$ constants c_1, c_2

Thm [Gabrielov-Vorobjov] consider the problem of [illegible] a semi-algebraic set $S \subseteq \mathbb{R}^n$. $\exists$ constants c_1, c_2

$$\text{height of ACT for this problem} \geq \frac{c_1 b_m(S)}{m+1} - c_2 n$$

m^{th} Singular Betti number of S.

- Complexity theory of Constructible sheaves [Basu]
- Categorical Complexity [Basu-Isik]
  define complexity of Categories & functors
  recovers classical notions in complexity theory.

"Rising Sea" approach in Complexity theory

The Rising Sea

Contents

The "rising sea" is a metaphor due to Alexander Grothendieck (see the quote below), meaning to illuminate how the development of general abstract theory eventually brings with it effortless solutions to concrete particular problems, much like a hard nut may be cracked not immediately by sheer punctual force, but eventually by gently immersing it into a whole body of water.

- Freedman

ANNALS OF MATHEMATICS

Complexity classes as mathematical axioms

By Michael H. Freedman

assumes stronger than $P \neq NP \implies$ Knots with certain properties exist

Lec 2 (Complexity of Matrix Mult.)

Wednesday, 10 May 2023 03:03

Complexity of M.M.

$$A_{n\times m} \cdot B_{m\times l} = C_{n\times l}$$

$$c_{i,j} = \sum_{k=1}^{m} a_{i,k} b_{k,j}$$

Naive method n mults and $n-1$ additions per dot product.
total of $O(n^3)$ operations.

Strassen

① You can do 2×2 multiplication using 7 products & 18 sums, instead of 8 products & 4 sums.

② Recurse to $n\times n$ mat. mult. using

$$O\left(n^{\log_2 7}\right) \text{ operations}$$

It is a fact (Prop 15.1 in Burgisser et. al) that total complexity is governed by the no. of multiplications.

Thus we can look at TENSOR RANK to count number of multiplications

$$\omega := \inf\{h \in \mathbb{R} \mid n\times n \text{ mat. mult. can be done using } O(n^h) \text{ operations}\}$$

$M_{\langle k,m,n\rangle} : \mathbb{C}^{k\times m} \times \mathbb{C}^{m\times n} \longrightarrow \mathbb{C}^{k\times n}$ is the mat. mult. map.

$M_{\langle k,m,n\rangle}$ is bilinear, so it is a tensor in $(\mathbb{C}^{k\times m})^* \otimes (\mathbb{C}^{m\times n})^* \otimes \mathbb{C}^{k\times m}$

Thm [Strassen]

$$\omega = \inf\{\tau \in \mathbb{R} \mid \overset{\text{Rank}}{R}(M_{\langle n,n,n\rangle}) = O(n^\tau)\}$$

N.B. ① In principle ω can depend on characteristic

② (Conjecture) $\omega = 2$

③ ... limit cannot be achieved

② (Conjecture) $\omega = 2$

③ ω is defined to be a limit pt. Limit cannot be achieved

Plan ① Explain Strassen's 2x2 alg "symmetrically"

② Use u.b on $R(M_{\langle k,m,n\rangle})$ for any specific k, m, n to turn it into u.b. on ω.

Strassen $R(M_{\langle 2,2,2\rangle}) \leq 7$ 2.81

Pan $R(M_{\langle 70,70,70\rangle}) \leq 143240$ 2.79

③ Border rank, and use u.b. on border rank to u.b. of ω. $\underline{R}(M_{\langle k,m,n\rangle})$

④ Schonhage γ-theorem

upperbounds on $R(\oplus M_{\langle * \rangle})$ to get u.b. on ω.

⑤ Coppersmith-Winograd

u.b. on $R((\oplus M_{\langle * \rangle})^{\boxtimes k})$ to get u.b. on ω

⑥ Cohn-Umans group theoretic approach.

Conceptual Strassen's alg

$[n] = \{1, \ldots, n\}$

$M_{\langle n\rangle} : M_n \times M_n \to M_n$

$M_{\langle n\rangle}$ as element of $M_n^* \otimes M_n^* \otimes M_n^*$, i.e.

$$M_{\langle n\rangle} = \sum_{i,j,k \in [n]} E_{i,j}^* \otimes E_{j,k}^* \otimes E_{k,i}^*$$

Observe that given $A \otimes B \otimes C \in M_n \otimes M_n \otimes M_n$

$$\langle M_{\langle n\rangle}, A \otimes B \otimes C \rangle = \text{trace}(ABC)$$

$$(AB)_{i,j} = \langle M_{\langle n\rangle}, A \otimes B \otimes E_{j,i} \rangle$$

$$(AB)_{i,j} = \langle M_{\langle n\rangle}, A \otimes B \otimes E_{j,i} \rangle$$

Notice [Symmetry] $X, Y, Z \in GL(n)$

$$\langle M_{\langle n\rangle}, (Z^{-1}AX) \otimes (X^{-1}BY) \otimes (Y^{-1}CZ) \rangle = \operatorname{trace}(ABC) \quad (*)$$

<u>fact</u> up to a constant, $M_{\langle n\rangle}$ is the only operator that has $(*)$ symmetry.

<u>Defn</u> A set S of n-dimensional vectors is a unitary 2-design if

$$\sum_{v \in S} v = 0 \quad \text{and} \quad \frac{1}{|S|} \sum_{v \in S} |v\rangle\langle v| = \frac{1}{n} I$$

Thm Let $S = \{w_1 \ldots w_s\}$ be a unitary 2-design. Then tensor rank of $M_{\langle n\rangle}$ is at most $s(s-1)(s-2)+1$ $\qquad A^{\otimes 3} \quad A \otimes A \otimes A$

(Lazy) <u>Proof</u> By defn

$\frac{s}{n} I$

(1) — $$\frac{s^3}{n^3} I^{\otimes 3} = \sum_{i,j,k \in [n]} |w_i\rangle\langle w_i| \otimes |w_j\rangle\langle w_j| \otimes |w_k\rangle\langle w_k|$$

(2) — $$\frac{s^3}{n^3} M_{\langle n\rangle} = \sum_{i,j,k \in [n]} |w_i\rangle\langle w_j| \otimes |w_j\rangle\langle w_k| \otimes |w_k\rangle\langle w_i|$$

M M

i,j	j,k	k,i
1,1	1,1	1,1
1,2	2,1	1,1
1,1	1,2	2,1
1 2	2 2	2,1
⋮		

I

i,i	j,j	k,k
1,1	1,1	1,1
1,1	1,1	2,2
1,1	2,2	1,1
1,1	2,2	2,2
⋮		

② − ① $\frac{s^3}{n^3}\left(M_{\langle n\rangle} - I^{\otimes 3}\right)$

$$= \sum_{\substack{i,j,k \\ \text{distinct}}} |w_i\rangle\langle w_j - w_i| \otimes |w_j\rangle\langle w_k - w_j| \otimes |w_k\rangle\langle w_i - w_k|$$

$M_{\langle n\rangle} = I^{\otimes 3} + \quad s(s-1)(s-2)$

$R(M_{\langle n\rangle}) \leq s(s-1)(s-2)$

$\boxtimes$

In $n=2$, the three corners of the equilateral triangle give a unitary 2-design

$S = \{ (1,0), (-1/2, \sqrt{3}/2), (-1/2, \sqrt{3}/2) \}$

$|S| = 3 \Rightarrow R(M_{\langle 2\rangle}) \leq 7$

$\boxtimes$

Prop If V is a non-trivial irrep of G (finite group), then $v \in V$ with $|v|^2 = 1$, the orbit of v is a unitary 2-design

Proof Schur's Lemma $\boxtimes$

Machinery to get u.b. of ω using u.b. of $R(M_{\langle k,m,n\rangle})$

Defn [Permutation of a tensor] Suppose $t = \sum_{i=1}^{r} t_j$, $t_j = a_{j,1} \otimes a_{j,2} \otimes a_{j,3}$.

for $\pi \in S_3$ define

$$\pi(t) = \sum_{i=1}^{r} \pi(t_j), \text{ where } \pi(t_j) = a_{j,\pi^{-1}(1)} \otimes a_{j,\pi^{-1}(2)} \otimes a_{j,\pi^{-1}(3)}$$

Check that $\pi(t)$ is well-defined ✓

Lemma $R(t) = R(\pi(t))$

Proof easy $\boxtimes$

Lemma

Proof easy $\boxtimes$

Defn Let $t \in A \otimes B \otimes C$. Let $f_1 : A \to A'$, $f_2 : B \to B'$, $f_3 : C \to C'$ be homomorphisms. $t = \sum_{i=1}^{r} a_{i,1} \otimes a_{i,2} \otimes a_{i,3}$

[RESTRICTION]

$$(f_1 \otimes f_2 \otimes f_3) \cdot t = \sum_{j=1}^{r} f_1(a_{i,1}) \otimes f_2(a_{i,2}) \otimes f_3(a_{i,3}) \in A' \otimes B' \otimes C'$$

Lemma $R((f_1 \otimes f_2 \otimes f_3)\, t) \le R(t)$, with equality if f_i's are isomorphisms

Proof easy $\boxtimes$

Lemma Take any $\sigma \in S_3$ $\quad R(M_{\sigma\langle k,m,n\rangle}) = R(M_{\langle k,m,n\rangle})$

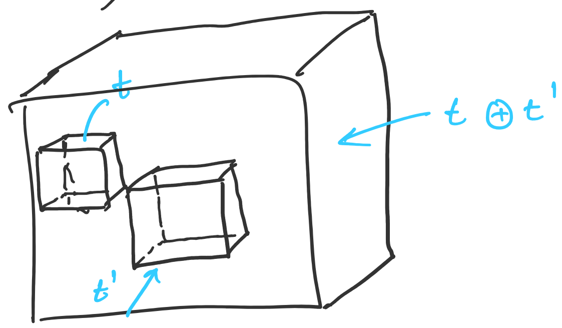
Proof

$$\begin{tikzcd} \mathbb{C}^{k\times m} \otimes \mathbb{C}^{m\times n} \arrow[d, leftrightarrow] \\ \mathbb{C}^{m\times n} \otimes \mathbb{C}^{k\times m} \arrow[d, "\text{transpose}"] \\ \mathbb{C}^{n\times m} \otimes \mathbb{C}^{m\times k} \end{tikzcd}$$

$\boxtimes$

Defn [Direct sum of tensors] $t \in \mathbb{F}^{k} \otimes \mathbb{F}^{m} \otimes \mathbb{F}^{n}$, $t' \in \mathbb{F}^{k'} \otimes \mathbb{F}^{m'} \otimes \mathbb{F}^{n'}$

$t \oplus t' \in \mathbb{F}^{k+k'} \otimes \mathbb{F}^{m+m'} \otimes \mathbb{F}^{n+n'}$



Lemma $R(t \oplus t') \le R(t) + R(t')$