Geometric Complexity Theory - contd...
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$DET_m = \overline{GL(V).\det_m}$, $PER^n_m = \overline{GL(V).\, z^{m-n} \text{perm}_n}$

Conjecture For $m = n^{O(1)}$,

$$z^{m-n}.\text{perm}_n \notin DET_m \iff PER^n_m \nsubseteq DET_m$$

Conjecture implies Valiant's conjecture

Today :-

We looked at the set of polynomials of wr ≤ 1.

e.g. $b^2 - 4ac$ is invariant under the SL_2 action $Sym^2(\mathbb{C}^2)$

$b^2 - 4ac = 0 \iff ax^2 + bxy + cy^2$ is a perfect square $\iff$ wr $= 1$

(*) $\text{Span}\{b^2 - 4ac\} \subseteq Sym^2\, Sym^2(\mathbb{C}^2)$ is a trivial rep of SL_2

Claim G rep V in the coord ring $\iff$ G-invariant property (Zariski-closed)

Another example

$GL_n \times GL_n$ acting on M_n

$$(A, B). M = A \times B^T$$

$$X = g . Y \iff \text{rank } X = \text{rank } Y$$
$$g \in GL_n \times GL_n$$

(*) Invariant ring is $\mathbb{C}[\det_n]$

Defn [Null cone] for a G-representation V, $N_G(V) \subseteq V$ is the null cone,

$N_G(V)$ — all pts whose orbit closures contain zero.

Defn [Null cone]

$N_G(V)$ – all pts whose orbit closures contain zero.

In our case, the null cone is the set of singular matrices

(*) There is an invariant to seperate orbits, i.e. rank, but rank is not a polynomial invariant

(*) However, since rank is a G-invariant, there is a G-inv. subspace in our coordinate ring that this corresponds to. MINORS!

$$\{ X \mid \operatorname{rank} X < r \}$$ is closed and is precisely cut out by the $r \times r$ minors.

(*) Thus the span of $r \times r$ minors is a G-representation

Abstractly :- If our space was $V \otimes W$ acted on by $GL(V) \times GL(W)$, the G-rep can be written

$$\Lambda^r V \otimes \Lambda^r W$$

Our co-ordinate ring is $\mathbb{C}[x_{ij}]$. $\Lambda^r V \otimes \Lambda^r W$ helps us exactly distinguish orbits

pts in the null cone look something this

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

Defn [Stable] x stable $\iff$ $G.x$ is closed

Thm [Matsushima] $\mathrm{Stab}_G\, x$ is a reductive subgroup.

Defn [Partial Stability] $[x] \in \mathbb{P}V$ is partially stable if $\exists$ maximal parabolic subgroup P of G.

[For GL_n diagonal matrices like $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$]

s.t $P \geq \mathrm{Stab}_G(x)$ and

① $\mathrm{Stab}_G(x) \geq R$ (unipotent radical of P) For GL_n $\begin{pmatrix} 0 & * \\ & 0 \end{pmatrix}$

② $[x]$ is stable w.r.t $K \leq L$ (Levi subgroup of G) $\begin{pmatrix} * & 0 \\ & * \end{pmatrix}$ GL_n

where K is reductive &

$$\operatorname{ran} K = \operatorname{rank} L - \delta$$ ← small think close to 0

and

$\mathrm{stab}_L\, x$ is reductive

③ $L.x \cong L/\mathrm{Stab}_L(x)$ ← we want this to be not too small relative to $G.x$

$$\frac{\dim L.x}{\dim G.x} \geq \Delta$$ ← think close to 1.

Remark if $P = G$, $\delta = 0$, $\Delta = 1$, this is equivalent to stability

(*) Partial stability refines information in the null cone.

Partial stability on matrices with rank

$GL(V) \times GL(W)$ acting on $V \otimes W$

$(A,B).X = AXB^T$

$$(A,B).\ X = A X B^T$$

☆ lower the rank, the "worse" S, Δ get.. Partial stab. quantifies how "deep" you are in the null cone.

Q. What is stabilizer of $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$?

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} B_{11}^T & B_{21}^T \\ B_{12}^T & B_{22}^T \end{pmatrix} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

$$A_{11} B_{11}^T = I_r \quad , \quad A_{11} B_{21}^T = A_{21} B_{11}^T = A_{21} B_{21}^T = 0$$

$\Rightarrow$ A_{11} & B_{11}^T are invertible, thus $A_{21} = B_{21} = 0$

Thus stabilizer

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \begin{pmatrix} A_{11}^{-T} & B_{12} \\ 0 & B_{22} \end{pmatrix} \longleftarrow$$

P for the Partial stab. Cond.

Cond 1 ✓ bcoz A_{11} & B_{12} are arbitrary.

Cond 2 take the action of $\begin{pmatrix} GL_r & \\ & GL_{n-r} \end{pmatrix}$ on $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$

$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$ is actually stable under the action

this is the inverse of ☆, not free.

$$\begin{pmatrix} SL_r^{☆} & 0 \\ 0 & GL_{n-r} \end{pmatrix} \times \begin{pmatrix} SL_r & 0 \\ 0 & GL_{n-r} \end{pmatrix}.$$ This is reductive, and

has rank one less than $\begin{pmatrix} GL_r & 0 \\ 0 & GL_{n-r} \end{pmatrix} \longleftarrow$ Levi subgroup effectively the stabilizer is

$$\begin{pmatrix} SL_r^{\leftarrow A_{11}} & \\ & GL_{n-r} \end{pmatrix} \quad \begin{pmatrix} A_{11}^{-T} & \\ & GL_{n-r} \end{pmatrix}$$

$$\dim L/_{stab_L(a)} = r^2 - 1$$ ← gets smaller as r gets smaller

Cond 3 ✓

UPCOMING:

→ Part. Stab. leads us to the notion of symmetries

Why are symmetries important?

Natural proofs [Razborov- Rudich]

P vs NP by showing $NP \nsubseteq P/Poly$

Thm [RR] Natural proofs cannot prove $NP \nsubseteq P/poly$

Defn A natural property of boolean functions is a subset

$$C_n \subseteq \text{fun}\left(\{0,1\}^n \to \{0,1\}\right)$$ that is :-

① Large

② Constructive (deciding if a func belongs to C_n can be done efficiently)

③ Useful against P/poly

↑ similar to separating polynomials.

A proof of $NP \nsubseteq P/poly$ has to violate ≥ one of the above three

Props

→ Cannot expect to violate ③

→ There is heuristic evidence that you cannot violate ②

→ Cannot expect to violate (3)
→ There is heuristic evidence that you cannot violate (2)

Stability & Partial stability

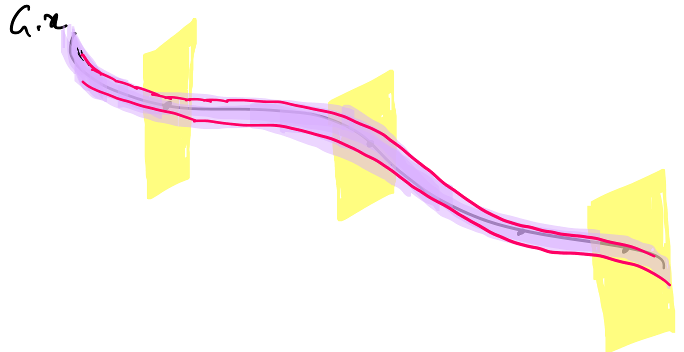
Thm [Luna's étale slice theorem] Let G act on V, $x \in V$ be stable. $H = \mathrm{Stab}_G(x)$

A neighbourhood of the orbit $G.x$ is (almost) isomorphic to $G \times_H N$.

normal vector space to the orbit

direct product, but elem of H can move about $(g, hn) \sim (hg, n)$

This means that if there is a pt y in the infinitesimal nbhd of Gx, then $\exists\, y' \in Gy$ s.t $\mathrm{stab}_G y \subseteq \mathrm{stab}_G x$



✱ IF you want to separate orbit closures, sufficient to look at stabilizers.

Pun: To understand a thm abt. groups, see it in action.

Claim $\mathrm{Perm}_n \notin \overline{GL_n \circ \det_n}$ ... "perm_n is far from the orbit of $\det_n$"

Proof FIRST PART

(1) $\mathrm{Stab}_G \mathrm{Perm}_n \sim (\mathbb{C}^\times)^{2n-1} \rtimes (S_n \times S_n) \rtimes \mathbb{Z}_2$

(2) $\mathrm{Stab}_G(\det_n) \sim S(GL_n \times GL_n) \rtimes \mathbb{Z}_2$

(A, B) s.t $\det(A)\det(B) = 1$

(3) Contradict suppose $\exists g$ st $\text{perm}_n = g \cdot \det_n$

$$\text{Stab}_G(\text{Perm}_n) = g\, \text{stab}_G \det_n g^{-1}$$

$\Rightarrow$ stabilizers need to be isomorphic.

$$\dim (\mathbb{C}^\times)^{2n-1} = 2n-1$$

$$\dim S(GL_n \times GL_n) = 2n^2 - 1$$

Stabilizers not even isomorphic !

SECOND PART:-

Luna's e'tale slice thm says because stabs are not even isomorphic, perm_n is not inside a nbhd of $\det_n$ $\boxtimes$

Partial stability gives you a weaker LES thm:

— If x is partially stable Gx is now a fiber bundle over a Grassmanian G/P with affine fibers. L/L'

Orbit $G.x$ looks like this:-

for every d- dim subspace ($\Uparrow$ Gr) ; you can pick a vector in L/L'

an by consequence get a pt. in the orbit.

$\Rightarrow$ The cohomology of $G.x$ is essentially the same as the Grassmanian.

"Partial stability implies the orbit has nice structure"

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* Perm & det are not just any random polynomials.

(*) They are characterized by their symmetries.

Defn G acts on V, $v \in V$ is char. by its symmetries

$\nearrow$ scalar

Defn G acts on V, $v \in V$ is char. by

if

$$\forall\, v' \text{ s.t } \mathrm{Stab}_G(v') \geq \mathrm{Stab}_G(v) \implies v' = \underset{\text{scalar}}{\lambda}\, v$$

Thm Both perm & det are characterized by their symm.

Sufficient cond for orbit closure separation

Defn G-group & two reps V & W, Let $\mathrm{Hom}_G(V, W)$ denote all $\phi : V \to W$ s.t $\forall g \in G$

$$\begin{array}{ccc} V & \xrightarrow{g} & V \\ \phi \downarrow & & \downarrow \phi \\ W & \xrightarrow{g} & W \end{array}$$

Commutes

Let $Z \subseteq \mathbb{C}^n$ be an alg. set such that $I(Z)$ is a homogeneous ideal. $\mathbb{C}[Z] = k[\bar{x}]/I(Z)$ has a grading

$$\mathbb{C}[Z]_\delta = k[\bar{x}]_\delta / I(Z)_\delta$$

$A := \overline{GL(V) \cdot z^{m-n}\, \mathrm{perm}_n}$, $A' := \overline{GL(V) \cdot \det_m}$. If it is true that for some m $A \subseteq A'$, then $I(A) \supseteq I(A')$

Consequently $I(A')_\delta \supseteq I(A)_\delta$

This gives a G-equiv surjection $\mathbb{C}[A']_\delta \twoheadrightarrow \mathbb{C}[A]_\delta$

$g \in GL(V)$ acts on $p(q(x))$

$$g \circ p(q(x)) = p(q(g^T x))$$

Defn For an irrep ρ of $GL(V)$, and another $GL(V)$ rep W

$$\text{mult}_\rho(W) = \dim \text{Hom}_{GL(V)}(\rho, W)$$

Let $W = U_1 \oplus \cdots \oplus U_t$

$$\dim \text{Hom}_{GL(V)}(\rho, W) = \left| \{ i \mid U_i \cong \rho \} \right|$$

By Schur's lemma, we have for all irreps ρ

is $\text{mult}_\rho(\mathbb{C}[A]) \leq \text{mult}_\rho(\mathbb{C}[A'])$

Thm Suppose there is an irrep ρ of $GL(V)$ s.t $\text{mult}_\rho(\mathbb{C}[A]) > \text{mult}_\rho(\mathbb{C}[A'])$, then

$$y^{m-n}\text{perm}_n \notin \overline{GL(V).\det_n}$$

$$\text{mult}_\rho(\text{Perm}) \gg \text{mult}_\rho \det = 0$$

P vs NP (NP vs P/Poly) in GCT

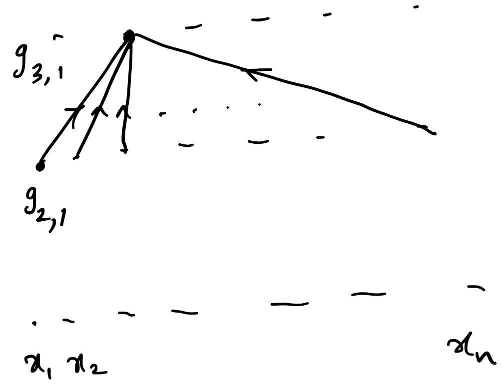
'H' – P-complete

'E' $\in$ NP but not known to be complete

H – Layered circuit 'n' inputs 'n' levels, 'n' gates in each level except the last.

• O/P

$$h_{\ell,i} = \sum_{j,k\in[n]} \alpha_{\ell,i,j,k}\, h_{\ell-1,j}\, h_{\ell-1,k}$$



Thm H_n is P/poly-complete over any finite field

Proof set α's to be anything $\boxtimes$

E: X_0 & X_1 - matrices over $\mathbb{F}_q$

$s = (s_1 \ldots s_n) \in \{0,1\}^n$, define X_s: ith column of X_s is the ith col. of X_{s_i}

$$E(X) = \prod_{s\in\{0,1\}^n} X_s$$

Thm Over $\mathbb{F}_q$, $E(X)^{q-1}$ is $\{0,1\}$-valued and is in NP.

Thm E & H are characterized by their symmetries

$\uparrow$ highly $\quad$ $\uparrow$ almost