Abhiram Natarajan : https://abhiramnatarajan.github.io : nataraj2@purdue.edu

Research Statement

1 Research Summary

I would describe my research leitmotif as *algebraic methods, taking inspiration from mathematical applications*. So far, I have worked along two directions.

1. Real-algebraic geometry and o-minimal geometry (both deterministic and random) toward applications in incidence combinatorics (Sections 1.1 and 1.2).

2. Homological algebra toward applications in computational complexity theory (Section 1.3).

Currently, I am looking at some interactions between commutative algebra and complexity theory (Section 2.1.4).

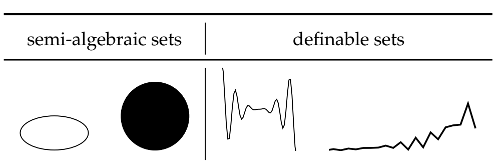
I am usually drawn to the pattern of hearing about an interaction between algebraic methods[1] and another mathematical area, and then working on the algebraic questions. In some cases, the algebraic motivation for my research has helped in contributing tools back to other areas which inspired the questions in the first place. I find this to be an exciting way to appreciate the vantage point that algebra affords.

1.1 Quantitative topological bounds in random semi-algebraic and o-minimal geometry

This section provides background, motivation and a description of joint work with Saugata Basu and Antonio Lerario [14].

1.1.1 Semi-algebraic and O-minimal geometry Real algebraic geometry is algebraic geometry over the real numbers $\mathbb{R}$, or more generally, over real closed fields. The primary focus of real algebraic geometry is semi-algebraic sets, defined as elements of the boolean algebra over sets of the form $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid P(x_1, \ldots, x_n) \leq 0\}$ for some $P \in \mathbb{R}[X_1, \ldots, X_n]$. Classical algebraic geometry, usually done over algebraically closed fields, enjoys the property that when an affine variety is projected down, the image is constructible. However, real varieties don't have this property; projections of real varieties can be semi-algebraic sets. Luckily, due to the Tarski-Seidenberg theorem, semi-algebraic sets project down to semi-algebraic sets too.

Semi-algebraic sets are known to possess many 'tameness' properties in a topological sense, such as stratifiability, finite traingulability, etc., which makes their study feasible. Grothendieck in his *Esquisse d'un Programme* [27] suggested that it would be important to - "...investigate classes of sets with the tame topological properties of semialgebraic sets...".



The answer to this is o-minimal geometry, introduced by Knight, Pillay and Steinhorn [42, 33, 43]. O-minimal geometry, whose genesis was in model theory, is an axiomatic generalization of semi-algebraic geometry, in so much as, many results about semi-algebraic sets are actually corollaries of results in o-minimal geometry. $\mathcal{S} = (S_n)_{n \in \mathbb{N}}$, with $S_n \subset \mathcal{P}(\mathbb{R}^n)$, is an o-minimal structure if it satisfies certain axioms. Elements of o-minimal structures are called definable sets, and definable sets often have some of the 'topological tameness' properties that semi-algebraic sets have.

The smallest structure containing all semi-algebraic sets is known to be o-minimal; however, the important point is that this is not the only one. There are now many more structures which have been proved to be o-minimal - e.g. smallest structure containing sets defined by the exponential function [52, 32], restricted analytic functions [49], Pfaffian functions [53], etc.

1.1.2 Quantitative bounds on topology of semi-algebraic and definable sets Quantitative bounds on the topology of sets arising in semi-algebraic geometry and o-minimal geometry have been studied for pure mathematical interest, because they shed light on the *topological complexity* of these sets, and, for effecting fundamental advances in real algebraic geometry, discrete geometry, statistical learning theory, convex optimization, algebraic complexity theory, etc. [9]. Given a semi-algebraic set $S \subset \mathbb{R}^n$, defined by at most m equations, each

[1] In the broad sense of including algebraic geometry, algebraic topology, commutative algebra, representation theory, etc.

of degree at most d, a prototypical topological question is – bound the Betti numbers of S in terms of m, d, n. The first results along this line were obtained by Oleinik and Petrovski [41], and later by Thom [47] and Milnor [38], who proved a bound of $O(d)^n$ on the sum of Betti numbers of any real algebraic variety in $\mathbb{R}^n$ defined by polynomials of degree at most d. This has been generalized to other types of semi-algebraic sets in several different ways, for e.g. [12, 11], and relatively more recently, e.g. [5, 6]. There has also been some work on Betti number bounds in o-minimal geometry, although the progress has not matched semi-algebraic geometry. For instance, [8] generalizes many quantitative bounds already known for semi-algebraic sets to the case of definable sets.

1.1.3 Departure point for our work One need for quantitative bounds from semi-algebraic geometry is in the *polynomial partitioning* theorem. Introduced in the seminal works of Guth and Katz [29, 28], the theorem has been critical in several types of problems in incidence geometry, e.g. [29, 46, 31]. It allows the decomposition of a given geometric arrangement into sub-arrangements of smaller size, thereby allowing a *divide-and-conquer* approach to the solution of a combinatorial problem. It continues to be critically important, to this day [3, 2], in resolving long-standing open problems in discrete geometry. This result strongly relies on the Bezout-type bounds of [5, 6].

Parallelly, incidences between definable sets over arbitrary o-minimal expansions of $\mathbb{R}$ has become an active research area as well [10, 21, 22]. The progress along this direction has been significantly slower though; each of these results use idiosyncratic techniques which don't really suggest methods of attack for other problems. One matter that has stymied progress is the unavailability of a polynomial partitioning type result for definable sets. Needless to say, an o-minimal polynomial partitioning theorem would enable progress on a lot of different fronts, and would potentially provide greatly simplified proofs of already proved results.

1.1.4 Pathologies in the intersection of definable hypersurfaces and algebraic varieties We now have the right context to summarize the main results in [14]. The primary ingredient required for o-minimal polynomial partitioning is control over the Betti numbers of the intersection of any fixed definable subset of $\mathbb{R}P^n$ with real algebraic hypersurfaces of growing degrees. Our first result gives a caution to this question. Given a sequence $\{Z_d\}_{d\in\mathbb{N}}$ of hypersurfaces, we construct a definable hypersurface $\Gamma \subset \mathbb{R}P^n$ containing a subset D homeomorphic to a disk, and a family of polynomials $\{p_m\}_{m\in\mathbb{N}}$ of degree $\deg(p_m) = d_m$ such that

$$(D, Z(p_m) \cap D) \sim (\mathbb{R}^{n-1}, Z_{d_m}), \tag{1}$$

i.e. the zero set of p_m in D is ambient-diffeotopic to Z_{d_m} in $\mathbb{R}^{n-1}$. In other words, the intersection of Γ with a hypersurface of degree d can be as complicated as we want. Also, we have that the intersection is transversal, thus rendering the *pathological intersection* stable under mild perturbations.

The above is in sharp contrast with the case when Γ is semialgebraic, where, by the results in [6], the homological complexity of the zero set of a polynomial p on Γ is bounded by a polynomial in $\deg(p)$. This result represents a dampener for hopes of a polynomial partitioning theorem on definable sets.

1.1.5 Random algebraic geometry Given a definable hypersurface Γ, we now know that there can be polynomials, such that Γ restricted to the zero set of the polynomial has pathological topology. Is it true that most polynomials have pathological intersection? Usually, what does the topology of the intersection look like? We give an answer to this question, only after introducing rigorous definitions of *most* and *usually*.

We consider a measure called the *Edelman-Kostlan* measure on the space of homogeneous polynomials of degree d in $n+1$ variables, defined by choosing each coefficient of $p = \sum_{|\alpha|=d} \xi_\alpha x_0^{\alpha_0} \cdots x_n^{\alpha_n}$ independently from a standard Gaussian distribution (i.e. $\xi_\alpha \sim \mathcal{N}\left(0, {}^{d!}/{}_{\alpha_0!\ldots\alpha_n!}\right)$). This measure is the restriction of the Fubini-Study measure to the space of real polynomials. The variances of the gaussian random variables ${}^{d!}/{}_{\alpha_0!\ldots\alpha_n!}$ are chosen in such a way that the resulting probability distribution is invariant under orthogonal change of variables (there are no preferred points or directions in $\mathbb{R}P^n$, where zeroes of p are naturally defined). Moreover, if we extend this probability distribution to the whole space of complex polynomials, by replacing *real* with *complex* Gaussian variables, it can be shown that this extension is the *unique* Gaussian measure which is invariant under unitary change of variables. This makes real Kostlan polynomials a natural object of study. This model for random polynomials received a lot of attention since the pioneering works of Edelman, Kostlan, Shub and Smale [23, 45, 44] on random polynomial systems solving.

1.1.6 Pathologies are rare Using morse theoretic arguments, we show that for any smooth compact definable Γ, for most polynomials a Bézout-type bound holds for the intersection $\Gamma \cap Z(p)$, i.e. for every $0 \leq k \leq n-2$ and $t > 0$:

$$\mathbb{P}\left[b_k(\Gamma \cap Z(p)) \geq td^{n-1}\right] \leq \frac{c_\Gamma}{td^{\frac{n-1}{2}}}, \tag{2}$$

for some constant c_Γ depending on the volume of Γ, where the probability is w.r.t. the Kostlan measure over the space of homogeneous polynomials of degree d in $n+1$ variables. On one hand, our first result seems to be a strong hindrance to have an o-minimal analogue of polynomial partitioning. However, on the other hand, we also prove that it is rare for pathological behavior to be exhibited. This gives some hope that a modified version of the technique can still be applicable to incidence questions. See Section 2.1.1 for how this result can be potentially used to obtain an o-minimal version of polynomial partitioning.

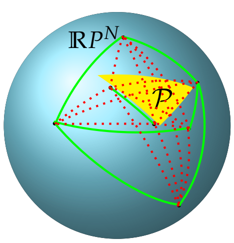
1.2 Betti numbers of random hypersurface arrangements

This section provides a summary of another joint work with Saugata Basu and Antonio Lerario [15].

1.2.1 Random hypersurface arrangements Continuing the study of the topology of random algebraic sets, we ask the following question - given random homogeneous polynomials $P_1, \ldots, P_s$, of degrees $d_1, \ldots, d_s$, in $n+1$ variables, what is the topological complexity of $\Gamma = \bigcup_{j=1}^{s} Z(P_j) \subset \mathbb{R}P^n$? Objects of this type appear quite naturally, and as explained before, partitioning problems is one example. Our first result gives information on the number of connected components of $\mathbb{R}P^n \backslash \Gamma$. Using a random spectral sequence argument, we show that

$$\mathbb{E}\left[b_0(\mathbb{R}P^n \setminus \Gamma)\right] = \sum_{\substack{I \subset [s] \\ |I| = n}} \sqrt{\prod_{i \in I} d_i} + O(\max\{d_1, \ldots, d_s\}^{(n-1)/2} s^{n-1}). \tag{3}$$

This study suggests a number of other questions regarding sign conditions on Kostlan hypersurface arrangements; see Section 2.1.2 for more.



1.2.2 Random quadrics arrangements Our next result deals instead with the asymptotic structure of Γ when $d_1, \ldots, d_s = 2$. This is the first non-trivial case, since for an arrangement of hyperplanes (i.e. with all degrees equal to one), the expected value of the topological complexity will coincide with that of deterministic generic arrangements. Since, it is known that the growth of the Betti numbers of semi-algebraic sets defined by quadratic polynomials show different behavior compared to that of general semi-algebraic sets (see e.g. [7, 13] for the deterministic case and [36, 37] in the random setting), it could be expected that the average topological complexity of arrangements consisting of quadric hypersurfaces would be smaller than in the general case (at least in the dependence on the number s of hypersurfaces).

It turns out that in the case of quadrics, the problem of understanding the number of connected components of Γ is related (via a theorem of Calabi [20]) to the connectivity of a certain random graph model, and can be studied in a precise way. The random graph model is as follows. We pick a semi-algebraic convex subset $\mathcal{P} \subset \mathbb{R}P^N$ and we sample independent points $q_1, \ldots, q_s$ from the uniform distribution on $\mathbb{R}P^N$. The vertices of the random graph are points $\{v_1, \ldots, v_s\}$ (one for each sample) and we put an edge between v_i and v_j, if and only if $i \neq j$ and the geodesic completion of the line connecting q_i and q_j does not intersect $\mathcal{P}$.

We prove a result which quantifies the expected number of connected components of this random graph model, which could be of independent interest. Using this, we show that in the case of quadrics

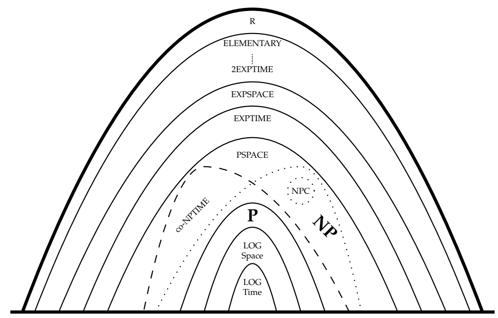
$$\lim_{s \to \infty} \frac{\mathbb{E}\left[b_0(\Gamma)\right]}{s} = 0, \tag{4}$$

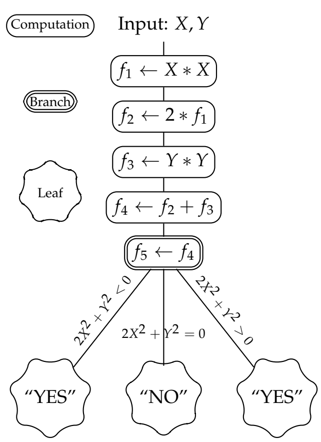
which is a tighter bound than what our more general result suggests.

1.3 Algebraic methods in complexity theory

The field of computational complexity theory attempts at classifying problems according to the amount of computational resources required to solve them [51]. While knowing how efficiently a problem can be computed is obviously a matter of utmost engineering importance, it also has deep philosophical implications [1]. Theorems in complexity theory can be regarded as 'quantitative epistemology'[2] in that they delineate the nature of knowledge using the language of mathematics. Naturally, complexity theory has shown interactions with nearly every area of pure mathematics - algebra, analysis, probability, logic, etc.

Some of the topics at the intersection of algebraic geometry, representation theory and computational complexity theory are the complexity of matrix multiplication, the VP != VNP conjecture[3] (see e.g. [35]), and algebraic decision and computation trees. Here, we shall focus on the latter two connections. We shall begin by describing some interactions of homological algebra with decision/computation tree lower bounds (Sections 1.3.1 and 1.3.2), as well as the VP $\neq$ VNP conjecture (Section 2.2.1). Finally, Section 2.1.4 deals with interactions between commutative algebra and complexity theory.





1.3.1 Topological invariants in complexity theory The algebraic computation tree is a model of computation. It is a representation of the computational steps that a Turing machine would execute, and it can be used for deciding if an input point belongs to a particular semialgebraic set. Obtaining a lower bound on the height of algebraic computation trees for a particular problem represents a fundamental barrier that no procedure can overcome. Lower bounds are usually notoriously difficult to prove; among the various viable methods for obtaining lower bounds, one of the most popular ones is where topological invariants such as the Euler characteristic and Betti numbers are used. This approach has a long history beginning from [16, 54], right up to the most recent result of Gabrielov and Vorobjov [24] who show that the height of an algebraic computation tree to decide membership in a semi-algebraic set $S \subset \mathbb{R}^n$ is bounded from below by $\frac{c_1 \log b_m(S)}{m+1} - c_2 n$, where $b_*(\cdot)$ are the singular Betti numbers.

1.3.2 Complexity of deciding tensor rank The rank of an arbitrary tensor $T \in \mathbb{F}^{\otimes_{i=1}^d n_i}$ is the minimum r such that T can be written as a sum of r rank-one tensors: $T = \sum_{\ell=1}^r \left(\bigotimes_{i=1}^d u_i^{(\ell)}\right)$. In ongoing work with Joshua Grochow, we are working on computing the cohomology rings of tensor spaces of fixed rank. For instance, we have that the cohomology ring of the space of tensors of rank exactly one (say $X \subseteq \mathbb{C}^{\otimes_{i=1}^d n_i}$) is

$$H^*(X, \mathbb{Z}) \cong \mathbb{Z}[X_1, \ldots, X_d] \Big/ \left\langle X_1^{n_1+1}, \ldots, X_d^{n_d+1}, \textstyle\sum_{i=1}^d X_i \right\rangle, \tag{5}$$

where each X_i has degree two. We have computed the cohomology of the space of tensors of rank exactly two, and are now working on computing the cohomology ring of the space of tensors of rank at most 2. See 2.1.3 for more discussion along this direction. Aside from providing important topological information, these results portend lower bounds on procedures to test tensor rank (c.f. Section 1.3.1).

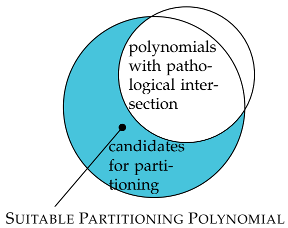
2 Current and Future Directions

In this section, I shall describe things that I am working on now (Section 2.1), as well as things that I am interested to work on in the future, both near and distant (Section 2.2).

[2]Words borrowed from Scott Aaronson.
[3]The algebraic version of the P $\neq$ NP conjecture.

2.1 Current work

Below I shall describe some of the immediate problems that I am working on. There certainly are obvious and incremental extensions of the works already described that can be made, as well as substantial difficult open problems. Here I shall focus only on the latter. As before, I continue to draw inspiration from interactions with incidence combinatorics (Sections 2.1.1 and 2.1.2), and computational complexity theory (Sections 2.1.3 and 2.1.4).



2.1.1 O-minimal polynomial partitioning With Saugata Basu and Antonio Lerario, we continue to investigate if a polynomial partitioning theorem for definable sets is feasible. We proved in [14] that for any definable hypersurface $\Gamma \subset \mathbb{R}P^n$, if p is a random Kostlan homogeneous polynomial of degree d in $n+1$ variables, $\mathbb{P}\left[b_0(\Gamma \cap Z(p)) \gtrsim d^n\right] \lesssim \frac{1}{\sqrt{d^n}}$. This means that the measure of bad polynomials grows smaller with increasing degree. If we are able to prove that the measure of the set of polynomials with properties desirable for partitioning is strictly $\gtrsim \frac{1}{\sqrt{d^n}}$, by the probabilistic method, we would have proved that there exists a polynomial which desirable properties, which also does not have pathological topological complexity on restriction to any definable hypersurface Γ, giving us the much needed o-minimal polynomial partitioning theorem.

2.1.2 Topology of random sign conditions In [15], we show that the expected total Betti number of $\mathbb{R}P^n \setminus \Gamma$ has the same order as that of the zeroth Betti number (Equation 3), suggesting a conjecture that a random cell, on average, is homotopy equivalent to a point. Also, for instance, if we had s Kostlan polynomials each of degree d, then our results show that there are on average $s^n d^{n/2}$ connected components in the complement of the zero set of the arrangement. In principle, there are 2^s sign conditions, which means that there could in principle be 2^s connected components in the complement of the zero set of the arrangement. Consequently, for certain values of s, d, it means that many of these 2^s conditions are unrealizable. Thus, one question is to study the probability of a sign condition on an arrangement of Kostlan polynomials to be realizable. Another question is to study the average Betti numbers of realizable sign conditions. This will have applications in partitioning random configurations.

2.1.3 Cohomology rings of tensors of fixed border rank With Joshua Grochow, we are continuing to compute the cohomology of various spaces of interest in computational complexity theory. Continuing our work on computing the cohomology of the space of tensors of fixed rank, we are now working on the space of tensors of rank at most 2. The rank 2 case crucially relies on a characterization of Kruskal [34]. As a first step, we are computing the de Rham cohomology of matrices of rank at most 2 (defined by the vanishing of all 3×3 determinants). Aside, using a particular spectral sequence of Gabrielov-Vorobjov-Zell [25], we believe we have a simple way of computing the cohomology ring of the symmetric square ($\mathrm{Sym}^2(\cdot)$) of various spaces, many of which appear to be currently unknown ([18] contains a discussion about the state-of-art). Naturally, after this, we expect to continue to study tensor spaces of higher rank.

2.1.4 Ulrich complexity Independently, I am currently looking at the work of Blaser et al. [17] where they propose a notion of complexity of polynomials and varieties called Ulrich complexity. For a homogeneous polynomial f, the ulrich complexity of f is the smallest r such that there exists a square matrix M of homogeneous linear forms with $\det M = f^r$, and there is a matrix N with $M \cdot N = f \cdot I$. They compute the Ulrich complexity for certain specific examples and conjecture that in general, it might be easier to compute than determinantal complexity. Valiant's VP != VNP conjecture [48] would imply exponential behavior of the Ulrich complexity of the permanent of an $n \times n$ matrix as well, and thus it is an open problem to prove that the Ulrich complexity of the n-permanent is at least exponential in n.

A more general open problem, which is a special case of the Buchweitz–Greuel–Schreyer conjecture [19], is to prove that the Ulrich complexity of any homogeneous f is bounded from below by $2^{\lceil s(f)/2 \rceil - 2}$, where $s(f)$ denotes the codimension of the singular locus of f. Also, they introduce interesting connections with commutative algebraic concepts. For $R = k[X_0, \ldots, X_n] \,/\, I$, F is defined to be an Ulrich module if F is a maximal Cohen-Macaulay R-module and also has a free resolution over $k[X_0, \ldots, X_n]$ whose differentials are matrices of linear forms. They extend the definition of Ulrich complexity to all projective varieties as the infimum of the rank of an Ulrich module on the variety. Another open problem is to actually construct the corresponding coherent sheaves, called Ulrich sheaves. This has been done on hypersurfaces [4], but it is unknown even for codimension 2.

Finally, it would also be interesting to study the expected Ulrich complexity of a Kostlan distributed polynomial. A Kostlan polynomial is non-singular with probability 1, so it is reasonable to expect that the expected Ulrich complexity should be bounded from above by a constant.

2.2 Future work

There are two properties that I think will characterize my research for the foreseeable future. First, any question that I study will likely have algebraic methods in a non-trivial way. Second, I would like to work on questions where an answer would have some *conceptual meaning*, i.e. something beyond shaving off a log factor with idiosyncratic techniques. I'd like to keep a balance between questions which are concrete and focused (Sections 2.2.1 and 2.2.2), and directions which are more exploratory (Sections 2.2.3 and 2.2.4). In this section, I will detail some of the larger plans that I have.

2.2.1 Cohomology rings of the determinantal variety, and $\Sigma\Pi\Sigma$ circuits In the same vein as our efforts to obtain the cohomology rings of the space of tensors of fixed border rank, it would be important to compute the topology of other spaces of interest as well. One such is space is the orbit closure of the $n \times n$ determinant $\det_n$, i.e $\overline{GL(n^2, \mathbb{F}).\det_n} \subset \mathrm{Sym}^n(\mathbb{F}^{n\times n})^*$. The geometric complexity theory program proposed by Mulmuley and Sohoni [39, 40] equates Valiant's VP $\neq$ VNP conjecture [48] to a representation-theoretic problem of separating the orbit closure of the determinant from the orbit closure of the permanent. The idea is to cast the problem in algebraic language with the hope that advanced tools in the fertile areas of algebraic geometry and representation theory will allow us to make progress on the problem. While separation is obviously an astronomically difficult task, computing the cohomology of these spaces would help us obtain a coarse understanding of the geometry of these spaces and thus be a definitive first step.

Another space of interest is the subspace of polynomials that can be represented in $\Sigma\Pi\Sigma$ form, i.e. the image of the map $\mathbb{F}^t \otimes \mathbb{F}^d \otimes \mathbb{F}^{n+1} \to \mathbb{F}[X_1, \ldots, X_n]_{(\leq d)}$, mapping

$$(c_{ijk})_{i\in[t],j\in[d],k\in[n+1]} \mapsto \sum_{i=1}^{t}\prod_{j=1}^{d}\left(c_{ij1} + \sum_{k=2}^{n+1} c_{ijk}x_{k-1}\right). \tag{6}$$

While these spaces have been actively studied, very little is known about their cohomology.

2.2.2 Complexity of the image of a polynomial map As suggested in this work by Grochow [26] - For a polynomial map $\phi : \mathbb{F}^n \to \mathbb{F}^m$, given parameters such as the degree, circuit size, or monomial sparsity of ϕ, one would like to know the 'complexity' of the image of ϕ in terms of the number of Zariski-closed sets required to describe it, the complexity of the ideals of those sets, and the complexity of the boolean combination itself. The image $\phi(\mathbb{F}^n)$ can obviously be described by the formula $\psi : \exists(x_1, \ldots, x_n)$ such that $F_1(x_1, \ldots, x_n, y_1, \ldots, y_m) = \ldots = F_m(x_1, \ldots, x_n, y_1, \ldots, y_m) = 0$. Let ψ' be the quantifier-free version of ψ. This question is tantamount to studying the complexity of ψ'. One way to approach this problem would be to recast quantifier elimination algorithms, e.g. [30], and provide guarantees in terms of the parameters of ψ.

2.2.3 Minimally encapsulating simple semi-algebraic set Given a semi-algebraic set $S \subset \Delta_n$ whose description is given, is there an algorithm to output another semi-algebraic set $S' \supset S$ such that the lebesgue measure of $S' \setminus S$ is minimized, and S' has 'constant description complexity', i.e. its description is given by a constant number of polynomials each with a constant number of monomials? In other words, given a semi-algebraic set, you want to find a candidate within a parametric family of semi-algebraic sets that contains the given semi-algebraic set and that minimizes the lebesgue measure of the difference.

This process generalizes many concentration of measure inequalities. For instance, the Markov's inequality defines a semi-algebraic set that minimally contains the semi-algebraic set $\{0 \leq p_1, \ldots, p_n \leq 1, \sum_{i=1}^{n} p_i = 1 \sum_{i=1}^{n} i \cdot p_i = \mu\}$. Thus an algorithm to solve this problem would potentially be an algorithm that can propose new concentration inequalities.

2.2.4 Singularity theory for statistical learning theory I am very curious to study the work of Watanabe [50] which introduces techniques from singularity theory into statistical learning. Many statistical models such as neural networks, hidden markov models, etc, are known to be singular, i.e. their Fisher information matrix (depends on the probability density and the parameter space) has determinant 0. The viewpoint presented in this book is that, using singularity theory concepts, better understanding, and consequently more accurate statistical estimation, can be achieved in these models.

References

[1] Scott Aaronson. Why philosophers should care about computational complexity. *Computability: Turing, Gödel, Church, and Beyond*, pages 261–328, 2013.

[2] Pankaj K. Agarwal, Boris Aronov, Esther Ezra, and Joshua Zahl. An efficient algorithm for generalized polynomial partitioning and its applications. In *35th International Symposium on Computational Geometry, SoCG 2019, June 18-21, 2019, Portland, Oregon, USA.*, 2019. doi: 10.4230/LIPIcs.SoCG.2019.5.

[3] Boris Aronov, Esther Ezra, and Joshua Zahl. Constructive polynomial partitioning for algebraic curves in $\mathbb{R}^3$ with applications. In *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 2636–2648. SIAM, 2019.

[4] Jörgen Backelin and Jürgen Herzog. On ulrich-modules over hypersurface rings. In *Commutative algebra*, pages 63–68. Springer, 1989.

[5] Sal Barone and Saugata Basu. Refined bounds on the number of connected components of sign conditions on a variety. *Discrete & Computational Geometry*, 47(3):577–597, 2012.

[6] Sal Barone and Saugata Basu. On a real analog of bezout inequality and the number of connected components of sign conditions. *Proceedings of the London Mathematical Society*, 112(1):115–145, 2016.

[7] A. I. Barvinok. On the Betti numbers of semialgebraic sets defined by few quadratic inequalities. *Math. Z.*, 225(2):231–244, 1997. ISSN 0025-5874.

[8] Saugata Basu. Combinatorial complexity in o-minimal geometry. *Proceedings of the London Mathematical Society*, 100(2):405–428, 2009.

[9] Saugata Basu. Algorithms in real algebraic geometry: a survey. In *Real algebraic geometry*, volume 51 of *Panor. Synthèses*, pages 107–153. Soc. Math. France, Paris, 2017.

[10] Saugata Basu and Orit E Raz. An o-minimal szemerédi–trotter theorem. *The Quarterly Journal of Mathematics*, 69(1):223–239, 2017.

[11] Saugata Basu, Richard Pollack, and Marie-Françoise Roy. On the number of cells defined by a family of polynomials on a variety. *Mathematika*, 43(1):120–126, 1996.

[12] Saugata Basu, Richard Pollack, and Marie-Françoise Roy. On the betti numbers of sign conditions. *Proceedings of the American Mathematical Society*, 133(4):965–974, 2005.

[13] Saugata Basu, Dmitrii V. Pasechnik, and Marie-Françoise Roy. Bounding the Betti numbers and computing the Euler-Poincaré characteristic of semi-algebraic sets defined by partly quadratic systems of polynomials. *J. Eur. Math. Soc. (JEMS)*, 12(2):529–553, 2010. ISSN 1435-9855. doi: 10.4171/JEMS/208.

[14] Saugata Basu, Antonio Lerario, and Abhiram Natarajan. Zeroes of polynomials on definable hypersurfaces: pathologies exist, but they are rare. *The Quarterly Journal of Mathematics*, 70(4):1397–1409, 10 2019. ISSN 0033-5606. doi: 10.1093/qmath/haz022. URL `https://doi.org/10.1093/qmath/haz022`.

[15] Saugata Basu, Antonio Lerario, and Abhiram Natarajan. Betti numbers of random hypersurface arrangements. *arXiv preprint arXiv:1911.13256*, 2019.

[16] Michael Ben-Or. Lower bounds for algebraic computation trees. In *Proceedings of the fifteenth annual ACM symposium on Theory of computing*, pages 80–86. ACM, 1983.

[17] Markus Bläser, David Eisenbud, and Frank-Olaf Schreyer. Ulrich complexity. *Differential Geometry and its Applications*, 55:128–145, 2017.

[18] Yumi Boote and Nigel Ray. On the symmetric squares of complex and quaternionic projective space. *Glasgow Mathematical Journal*, 60(3):703–729, 2018.

[19] R-O Buchweitz, G-M Greuel, and F-O Schreyer. Cohen-macaulay modules on hypersurface singularities ii. *Inventiones mathematicae*, 88(1):165–182, 1987.

[20] Eugenio Calabi. Linear systems of real quadratic forms. *Proc. Amer. Math. Soc.*, 15:844–846, 1964. ISSN 0002-9939. doi: 10.2307/2034611.

[21] Artem Chernikov and Sergei Starchenko. Regularity lemma for distal structures. *J. Eur. Math. Soc. (JEMS)*, 20(10):2437–2466, 2018. ISSN 1435-9855. doi: 10.4171/JEMS/816.

[22] Artem Chernikov, Sergei Starchenko, and Margaret EM Thomas. Ramsey growth in some nip structures. *Journal of the Institute of Mathematics of Jussieu*, pages 1–29, 2016.

[23] Alan Edelman and Eric Kostlan. How many zeros of a random polynomial are real? *Bull. Amer. Math. Soc. (N.S.)*, 32(1):1–37, 1995. ISSN 0273-0979. doi: 10.1090/S0273-0979-1995-00571-9.

[24] Andrei Gabrielov and Nicolai Vorobjov. On topological lower bounds for algebraic computation trees. *Foundations of Computational Mathematics*, 17(1):61–72, 2017.

[25] Andrei Gabrielov, Nikolai Vorobjov, and Thierry Zell. Betti numbers of semialgebraic and sub-pfaffian sets. *Journal of the London Mathematical Society*, 69(1):27–43, 2004.

[26] Joshua A Grochow. Unifying known lower bounds via geometric complexity theory. *computational complexity*, 24(2):393–475, 2015.

[27] Alexandre Grothendieck. Esquisse d'un programme. *London Mathematical Society Lecture Note Series*, pages 5–48, 1997.

[28] Larry Guth. Polynomial partitioning for a set of varieties. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 159, pages 459–469. Cambridge University Press, 2015.

[29] Larry Guth and Nets Hawk Katz. On the erdős distinct distances problem in the plane. *Annals of Mathematics*, pages 155–190, 2015.

[30] Joos Heintz. Definability and fast quantifier elimination in algebraically closed fields. *Theoretical Computer Science*, 24(3):239–277, 1983.

[31] Haim Kaplan, Jiří Matoušek, and Micha Sharir. Simple proofs of classical theorems in discrete geometry via the guth–katz polynomial partitioning technique. *Discrete & Computational Geometry*, 48(3):499–517, 2012.

[32] AG Khovanskiĭ. *Fewnomials*, volume 88. American Mathematical Soc., 1991.

[33] Julia F Knight, Anand Pillay, and Charles Steinhorn. Definable sets in ordered structures. ii. *Transactions of the American Mathematical Society*, 295(2):593–605, 1986.

[34] Joseph B Kruskal. Three-way arrays: rank and uniqueness of trilinear decompositions, with application to arithmetic complexity and statistics. *Linear algebra and its applications*, 18(2):95–138, 1977.

[35] Joseph M Landsberg. *Geometry and complexity theory*, volume 169. Cambridge University Press, 2017.

[36] Antonio Lerario. Random matrices and the average topology of the intersection of two quadrics. *Proc. Amer. Math. Soc.*, 143(8):3239–3251, 2015. ISSN 0002-9939. doi: 10.1090/proc/12324. URL `http://dx.doi.org/10.1090/proc/12324`.

[37] Antonio Lerario and Erik Lundberg. Gap probabilities and betti numbers of a random intersection of quadrics. *Discrete & Computational Geometry*, 55(2):462–496, 2016.

[38] John Milnor. On the betti numbers of real varieties. *Proceedings of the American Mathematical Society*, 15(2): 275–280, 1964.

[39] Ketan D Mulmuley. On p vs. np and geometric complexity theory: Dedicated to sri ramakrishna. *Journal of the ACM (JACM)*, 58(2):5, 2011.

[40] Ketan D Mulmuley and Milind Sohoni. Geometric complexity theory i: An approach to the p vs. np and related problems. *SIAM Journal on Computing*, 31(2):496–526, 2001.

[41] O Oleinik and I Petrovsky. On the topology of real algebraic hypersurfaces. *Izv. Acad. Nauk SSSR*, 13: 389–402, 1949.

[42] Anand Pillay and Charles Steinhorn. Definable sets in ordered structures. i. *Transactions of the American Mathematical Society*, 295(2):565–592, 1986.

[43] Anand Pillay and Charles Steinhorn. Definable sets in ordered structures. iii. *Transactions of the American Mathematical Society*, 309(2):469–476, 1988.

[44] Michael Shub and Steve Smale. Complexity of Bézout's theorem. I. Geometric aspects. *J. Amer. Math. Soc.*, 6(2):459–501, 1993. ISSN 0894-0347. doi: 10.2307/2152805.

[45] Michael Shub and Steve Smale. Complexity of Bezout's theorem. II. Volumes and probabilities. In *Computational algebraic geometry (Nice, 1992)*, volume 109 of *Progr. Math.*, pages 267–285. Birkhäuser Boston, Boston, MA, 1993. doi: 10.1007/978-1-4612-2752-6_19.

[46] József Solymosi and Terence Tao. An incidence theorem in higher dimensions. *Discrete & Computational Geometry*, 48(2):255–280, 2012.

[47] René Thom. Sur l'homologie des variétés algébriques réelles. *Differential and combinatorial topology*, pages 255–265, 1965.

[48] Leslie G Valiant. Completeness classes in algebra. In *Proceedings of the eleventh annual ACM symposium on Theory of computing*, pages 249–261. ACM, 1979.

[49] Lou Van den Dries. A generalization of the tarski-seidenberg theorem, and some nondefinability results. *Bulletin of the American Mathematical Society*, 15(2):189–193, 1986.

[50] Sumio Watanabe. *Algebraic geometry and statistical learning theory*, volume 25. Cambridge University Press, 2009.

[51] Avi Wigderson. P, np and mathematics–a computational complexity perspective. In *Proc. of the 2006 International Congress of Mathematicians*, volume 3, 2006.

[52] Alex J Wilkie. Model completeness results for expansions of the ordered field of real numbers by restricted pfaffian functions and the exponential function. *Journal of the American Mathematical Society*, 9(4):1051–1094, 1996.

[53] Alex J Wilkie. A theorem of the complement and some new o-minimal structures. *Selecta Mathematica, New Series*, 5(4):397–421, 1999.

[54] Andrew Chi-Chih Yao. Decision tree complexity and betti numbers. *Journal of computer and system sciences*, 55(1):36–43, 1997.