Betti Numbers of Random Hypersurface Arrangements

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Complexity of Arrangements

- Goal: Betti numbers of arrangements of algebraic sets, i.e. $\bigcup_{i=1}^{s} Z(P_i)$ - important research area with applications
- Previous work: $P_1, \ldots, P_s \in \mathbb{R}[X_1, \ldots, X_n]$, max degree d:

Sum of Betti nos.	Individual Betti numbers
$\sum_{j \geqslant 0} b_j \left(\bigcup_{i=1}^{s} Z(P_i)\right) = O(s^n d^n)$	$b_j \left(\bigcup_{i=1}^{s} Z(P_i)\right) = s^{n-j} O(d)^n$

Question: What are the expected Betti numbers of an arrangement of random polynomials?

Distribution on Space of Polynomials

- *Gaussian* measure on $\mathbb{R}[X_0, \ldots, X_n]_{(d)}$ called Edelman-Kostlan measure: $P \sim \mathrm{KOS}(n, d)$ if
$$P(X_0, \ldots, X_n) = \sum_{\substack{\alpha=(\alpha_0,\ldots,\alpha_n)\\ \sum_{i=0}^{n} \alpha_i = d}} \xi_\alpha x_0^{\alpha_0} \ldots x_n^{\alpha_n},$$
where $\xi_\alpha \sim \mathcal{N}\left(0, \frac{d!}{\alpha_0! \ldots \alpha_n!}\right)$ are independent
- Orthogonally-invariance: for any $L \in O(n+1, \mathbb{R})$,
$$P(\mathrm{X}) \equiv_{\text{dist.}} P(L\mathrm{X})$$
- No points or directions are preferred in projective space

Expected Topology of Random Arrangements

Theorem (Basu-Lerario-N): Let $P_1, \ldots, P_s \in \mathbb{R}[X_0, \ldots, X_n]$ be homogeneous Kostlan forms, $\deg(P_i) \leqslant d$, and $\Gamma = \bigcup_{i=1}^{s} Z(P_i)$. Then
$$\mathbb{E}\left[b_0(\mathbb{RP}^n \setminus \Gamma)\right] = 2 s^n d^{n/2} + O\left(s^{n-1} d^{(n-1)/2}\right).$$
Also, for $0 < i \leqslant n-1$
$$\mathbb{E}\left[b_i(\mathbb{RP}^n \setminus \Gamma)\right] = O\left(s^{n-i} d^{(n-1)/2}\right).$$
Interpretation: Worst-case bound on b_0 is $\binom{s}{n} O(d^n)$, while expectation is equal to $2 s^n d^{n/2}$.

Proof - Random Mayer-Vietoris Spectral Sequence

- $A_1, \ldots, A_s$ - triangulations of $\Gamma_1, \ldots, \Gamma_s$, respectively
- $A_{\alpha_0,\ldots,\alpha_p} := \bigcap_{i=0}^{p} A_{\alpha_i}$; $C^i(A)$ - i-co-chains of A

Theorem: There exists a first quadrant cohomological spectral sequence converging to the cohomology of the union $(E_r, \delta_r)_{r \in Z}$:
$$E_r = \bigoplus_{p,q \in \mathbb{Z}} E_r^{p,q}, \quad \text{and} \quad E_0^{p,q} = \bigoplus_{\alpha_0 < \ldots < \alpha_p} C^q(A_{\alpha_0,\ldots,\alpha_p}),$$
with
$$\delta_r : E_r^{p,q} \to E_r^{p+r, q-r+1}, \qquad E_{r+1} \cong H_{\delta_r}(E_r).$$
Proposition: Define $e_r^{a,b} := \mathbb{E}\left[\text{rank } E_r^{a,b}\right]$. $e_{r+1}^{p,q} \leqslant e_r^{p,q}$, and, if $E_r^{p+r,q-r+1} = 0$, $e_{r+1}^{p,q} \geqslant e_r^{p,q} - e_r^{p-r,q+r-1}$.

Betti Numbers of Sets Defined by Quadrics

Growth of Betti numbers of s.a. sets defined by quadratic polynomials often shows behaviour different to general semi-algebraic sets. What is the expected Betti number of a union of random quadrics?

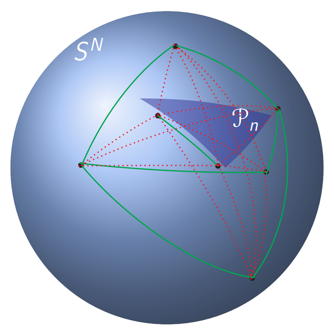
Theorem (Basu-Lerario-N): Let $P_1, \ldots, P_s \in \mathbb{R}[X_0, \ldots, X_n]$ be homogeneous Kostlan quadrics. Define $\Gamma = \bigcup_{i=1}^{s} Z(P_i)$. Then
$$\lim_{s \to \infty} \frac{\mathbb{E}\left[b_0(\Gamma)\right]}{s} = 0.$$
Interpration: Our general theorem suggests $\mathbb{E}\left[b_0(\Gamma)\right] = O(s)$. For quadrics, we prove $\mathbb{E}\left[b_0(\Gamma)\right] = o(s)$.

Equivalence to Random Graph

- $\mathrm{Sym}(n+1, \mathbb{R}) \cong \mathbb{R}[x_0, \ldots, x_n]_{(2)}, \qquad Q \mapsto \langle x, Qx \rangle$
- $\mathbb{RP}^N = \mathbb{P}(\mathrm{Sym}(n+1, \mathbb{R}))$, turns out sampling a Kostlan quadric is equivalent to sampling uniformly at random from S^N

Theorem (Calabi): Let $q_1, q_2 \in \mathbb{R}[x_0, \ldots, x_n]_{(2)}$ and $\Gamma_i = Z(q_i)$. Let $\mathcal{P}_n \subseteq S^N$ be the set of positive quadratic forms. Let $\ell \subset S^N$ be the projective line $\ell = \{[\lambda_1 q_1 + \lambda_2 q_2]\}_{\lambda_i \in \mathbb{RP}^1}$. Then:
$$\Gamma_1 \cap \Gamma_2 \neq \emptyset \iff \ell \cap \mathcal{P}_n = \emptyset.$$
Interpretation: Sampling process is equivalent to a random graph: Sample s points uniformly at random from S^N. Join points iff the great circle joining points does not pass through $\mathcal{P}_n$.

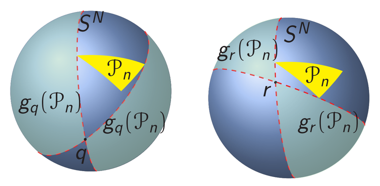
Illustration of Random Graph



Question: What is the average number of connected components in the above random graph denoted $\mathcal{G}(N, \mathcal{P}_n, s)$?

Obstacle Random Graph - Properties

- Good cone: for $q \in S^N$
$$g_q(\mathcal{P}_n) = \left\{x \in S^N \mid \ell(q, x) \cap \mathcal{P}_n = \emptyset\right\}.$$

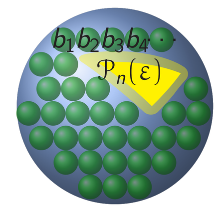


- Has flavour of $G_{n,p}$, but p is a random variable
$$\mathbb{P}\left[q' \text{ gets connected to } q\right] = \frac{\mathrm{vol}\,(g_q)}{\mathrm{vol}\,(S^N)}$$
- Probability random variables are not independent

Average Connected Components

Theorem (Basu-Lerario-N): The expected number of connected component of $\mathcal{G}(N, \mathcal{P}_n, s)$ satisfies:
$$\lim_{s \to \infty} \frac{\mathbb{E}\left[b_0(\mathcal{G}(N, \mathcal{P}_n, s))\right]}{s} \leqslant \frac{\mathrm{vol}\,(\mathcal{P}_n)}{\mathrm{vol}\,(\mathbb{RP}^N)}.$$
Interpretation: Considering $\frac{\mathrm{vol}(\mathcal{P}_n)}{\mathrm{vol}(S^N)}$ to be fixed, we have that the expected number of connected components is $o(s)$.

Average Connected Components - Proof



- For any $b_i \subseteq \mathcal{P}_n(\varepsilon)^c$, there exists $G_i \subseteq \mathcal{P}_n(\varepsilon)^c$,
$$\mu(G_i) > 0 \qquad \textit{and} \qquad \forall p \in G_i,\ g_p(\mathcal{P}_n) \supseteq b_i.$$
- Using coupon-collector type argument, bound number of samples required to collect all b_i. ∎

Ramsey-Theoretic Result

Corollary (Basu-Lerario-N): Let Γ be the graph of s quadrics. Then, for any $\varepsilon > 0$,
$$\lim_{s \to \infty} \mathbb{P}\left[\Gamma^c \text{ contains a clique of size } \varepsilon s\right] = 0.$$
Theorem (Alon-Pach-et-al.) For any semi-algebraic graph $G = (V, E)$, there exists a constant $\delta > 0$, such that one of the following is true:

1. There exists a clique of size $|V|^\delta$ in G.
2. The complement of G has a clique of size $|V|^\delta$.

Interpretation: Large cliques are impossible in Γ^c.