Partitioning Theorems for Sets of Semi-Pfaffian Sets, with Applications

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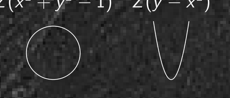
(based on joint work with Martin Lotz, Nicolai Vorobjov)

Real Algebraic Geometry Preliminaries

- Real Algebraic Set: The locus of common real zeros of $\{P_1, \ldots, P_s\} \subseteq \mathbb{R}[X_1, \ldots, X_n]$, i.e.,

$$Z(P_1, \ldots, P_s) := \{\mathrm{x} \in \mathbb{R}^n \mid P_1(\mathrm{x}) = \ldots = P_s(\mathrm{x}) = 0\}$$



- Semialgebraic set: A set $S \subseteq \mathbb{R}^n$ that is a finite Boolean combination of sets defined by polynomial inequalities:

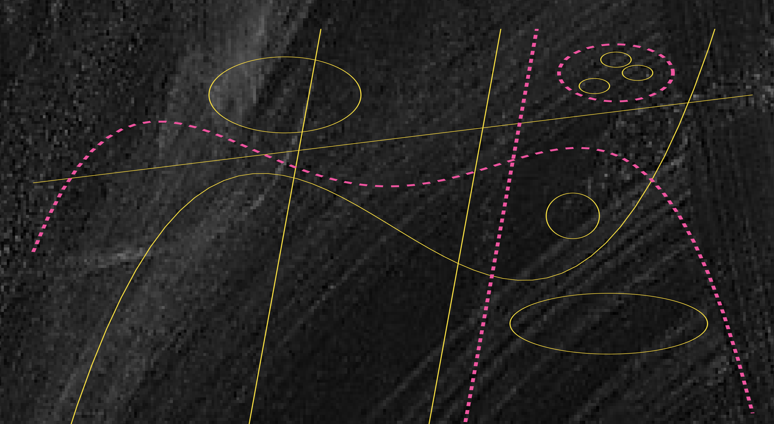
$$\{\mathrm{x} \in \mathbb{R}^n \mid P(\mathrm{x}) \geq 0\}$$

Polynomial Method in Combinatorics

- Algebro-geometric techniques have been very effective in incidence combinatorics [Sheffer, 2022]
- One technique, called polynomial partitioning, has helped solve several problems in incidence combinatorics, computational geometry, harmonic analysis, etc.

Theorem (Polynomial Partitioning [Guth and Katz, 2015, Guth, 2015])

Let Γ be a finite set of k-dimensional semialgebraic sets in $\mathbb{R}^n$. For any $D \geq 1$, there is a nonzero polynomial $P \in \mathbb{R}[X_1, \ldots, X_n]$ of degree $\leq D$, such that each commented component of $\mathbb{R}^n \setminus Z(P)$ intersects at most $\sim \frac{|\Gamma|}{D^{n-k}}$ algebraic sets of Γ.



Takeaway

There exists a general and flexible technique, that works in any dimension, to break space into simpler pieces allowing divide-and-conquer type approaches to problem solving.

Understanding Polynomial Partitioning

Let $P \in \mathbb{R}[X_1, \ldots, X_n]$ be of degree at most D:

- $\mathbb{R}^n \setminus Z(P)$ has at most $\sim D^n$ connected components (Oleinik-Petrovsky [1949], Milnor [1964], Thom [1965])
- A k-dimensional semialgebraic set $\gamma \in \Gamma$ intersects at most $\sim D^k$ connected components of $\mathbb{R}^n \setminus Z(P)$ (Barone-Basu [2012])
- We have $|\Gamma|$ no. of algebraic sets, so there are at most $\sim |\Gamma| \times D^k$ intersections
- $R^n \setminus Z(P)$ has at most D^n connected components, so $\sim \frac{|\Gamma| \times D^k}{D^n}$ denotes *equipartition*

See survey by Kaplan et al. [2012] for a wide range of applications in discrete geometry - e.g. Szemerédi-Trotter-type theorems, Counting Joints, Distinct Distances, Unit Distances, Cycle Elimination

O-minimal Geometry

- Semi-algebraic sets possess tameness properties such as stratifiability, triangulability, etc.
- *...investigate classes of sets with the tame topological properties of semialgebraic sets...* - Grothendieck, *Esquisse d'un Programme* [Grothendieck, 1997]
- O-minimal geometry (geometry of definable sets) is an axiomatic generalization of semi-algebraic geometry (Dries 1998)
- Semi-algebraic sets in $\mathbb{R}^n$ form an o-minimal structure
- Other examples - $\mathbb{R}$ with $\exp$ function (e.g. $x^3 + e^{x+2y} \leq 0$), Pfaffian functions (e.g. $x^\pi - e^{e^y} \leq \tanh(x)$)
- O-minimal incidence combinatorics is not as developed as algebraic incidence combinatorics

Question

Can we generalize polynomial partitioning to the o-minimal setting, i.e. with definable sets? well... we make progress...

Pfaffian functions

- Let $\mathcal{U} \subseteq \mathbb{R}^n$ be an open set. $\vec{q} = (q_1, \ldots, q_r)$, $q_i \in C^\infty(\mathcal{U})$, is a Pfaffian chain of chain-degree α and length r if there exist $P_{i,j} \in \mathbb{R}[X_1, \ldots, X_n, Y_1, \ldots, Y_i]_{(\leq \alpha)}$ such that
$$\frac{\partial q_i}{\partial x_j} = P_{i,j}(\mathrm{x}, q_1(\mathrm{x}), \ldots, q_i(\mathrm{x})).$$
- A function $g : \mathcal{U} \to \mathbb{R}$ is called a Pfaffian function w.r.t. $\vec{q}$ if there exists $Q \in \mathbb{R}[X_1, \ldots, X_n, Y_1, \ldots, Y_r]_{(\leq \beta)}$ such that
$$g(\mathrm{x}) = Q(\mathrm{x}, q_1(\mathrm{x}), \ldots, q_r(\mathrm{x})).$$
- (α, β, r) is called the format of g; β is called the degree of g.
- Zero locus of such a g is called a Pfaffian set. Locus of inequalities of Pfaffian functions is called a Semi-Pfaffian set.

Examples of Pfaffian Functions

- A polynomial of degree D is a Pfaffian function w.r.t. the empty chain; format is $(\alpha, D, 0)$ for any integer $\alpha > 0$.
- $\vec{q} = (q_1, \ldots, q_r)$, where $q_i(x) = e^{q_{i-1}(x)}$, and $q_0(x) = ax$, is a Pfaffian chain of order r and chain-degree r. Consequently, any $P \in \mathbb{R}[X, e^{aX}, e^{e^{aX}}, \ldots]$ is a Pfaffian function w.r.t. $\vec{q}$.
- $\vec{q} = (\tan(x))$ is a Pfaffian chain of order 1 and chain-degree 2 in the domain $\bigcap_{k \in \mathbb{Z}}\{x \in \mathbb{R} : x \neq \pi/2 + k\pi\}$, given $\mathrm{d}\tan(x) = (1 + \tan(x)^2)\ \mathrm{d}x$; any $P \in \mathbb{R}[X, \tan(X)]$ is a Pfaffian function w.r.t. $\vec{q}$.
- $\vec{q} = \left(\frac{1}{x}, \ln(x)\right)$ is a Pfaffian chain on the domain $R \setminus \{0\}$; any $P \in \mathbb{R}\left[X, \frac{1}{X}, \ln(X)\right]$ is a Pfaffian function.
- $\vec{q} = \left(\frac{1}{x}, x^m\right)$ for any $m \in \mathbb{R}$ is a Pfaffian chain; any $P \in \mathbb{R}\left[X, \frac{1}{X}, X^m\right]$ is a Pfaffian function w.r.t. $\vec{q}$.

The Pfaffian structure, i.e., the smallest collection of sets containing all semi-Pfaffian sets and that is stable under all structure operations, is an o-minimal structure.

Partitioning of Semi-Pfaffian Sets

Theorem (Polynomial and Pfaffian partitioning of semi-Pfaffian sets [Lotz et al., 2024])

Let Γ be a collection of semi-Pfaffian sets in $\mathbb{R}^n$ of dimension $k (\geq 1)$, where each $\gamma \in \Gamma$ is defined by at most m Pfaffian functions with format (α, β, r).

1. *For any $D \geq 1$, there is a non-zero polynomial $P \in \mathbb{R}[X_1, \ldots, X_n]$ of degree at most D, and a constant $C = C(n, m, \alpha, \beta, r)$, such that each connected component of $\mathbb{R}^n \setminus Z(P)$ intersects at most $C\frac{|\Gamma|}{D^{n-k-r}}$ elements of Γ.*
2. *Suppose that $\vec{q}$ is an algebraically independent Pfaffian chain of chain-degree α and order r, and suppose that all Pfaffian functions involved in defining the elements of Γ are defined w.r.t. $\vec{q}$. For any $D \geq 1$, there is a non-zero Pfaffian function P' with format (α, D, r) defined w.r.t. $\vec{q}$, and a constant $C = C(n, m, \alpha, \beta, r)$, such that each connected component of $\mathbb{R}^n \setminus Z(P')$ intersects at most $C\frac{|\Gamma|}{D^{n-k}}$ elements of Γ.*

Takeaway

1. *Generalization of Polynomial Partitioning to semi-Pfaffian sets,*
2. *New technique of Pfaffian Partitioning; guarantees are independent of r.*

Proof Details - 1

- A key step for o-minimal polynomial partitioning is, given a k-dimensional definable set $\gamma \subseteq \mathbb{R}^n$, for any polynomial $P \in \mathbb{R}[X_1, \ldots, X_n]$ of degree at most D, we would like an upper bound on the number of connected components of $\gamma \cap (\mathbb{R}^n \setminus Z(P))$.
- A result of Basu et al. [2019] suggests that a uniform bound might not be possible for definable sets in any arbitrary o-minimal structure: for every sequence of $(a_d)_{d \in \mathbb{N}}$, there is a regular compact and definable hypersurface $\gamma^* \subseteq \mathbb{RP}^n$, a subsequence $(a_{d_m})_{m \in \mathbb{N}}$, and a sequence of homogeneous polynomials $(P_m)_{m \in \mathbb{N}}$ of degrees $(d_m)_{m \in \mathbb{N}}$, such that
$$b_k(\gamma^* \cap Z(P_m)) \geq a_{d_m},$$
where b_k is the k^{th} singular Betti number; see [Natarajan, 2020, Section 3.4] for more discussion.

Takeaway

You can make the Betti numbers of the intersection of a definable hypersurface and an algebraic set arbitrarily large.

Proof Details - 2

Theorem (Semi-Pfaffian Bézout type bound [Lotz et al., 2024])

Let $\vec{q}$ be a Pfaffian chain of chain-degree α and order r. Let $\gamma \subseteq \mathbb{R}^n$ be a k-dimensional semi-Pfaffian set defined by at most m Pfaffian functions of degree at most β w.r.t. $\vec{q}$. Then for any Pfaffian function P w.r.t. $\vec{q}$ of degree D, there exists a constant $C = C(n, m, \alpha, \beta, r)$ such that

$$b_0\left(\gamma \cap (\mathbb{R}^n \setminus Z(P))\right) \leq CD^{k+r}.$$

- The above bound is obtained by setting up a system of Pfaffian equations such that the number of zeros of the system gives an upper bound on the number of connected components of $\gamma \cap (\mathbb{R}^n \setminus Z(P))$.
- This requires a weak stratification of γ [Gabrielov and Vorobjov, 2004], and then counting the number of solutions of the above system using Khovanskiĭ's Bézout type bound [Khovanskiĭ, 1991, §3.12, Corollary 5] for Pfaffian sets.

Application: Pfaffian Szemerédi-Trotter

Given a set $\mathcal{P}$ of points in $\mathbb{R}^n$, and a set Γ of subsets of $\mathbb{R}^n$, the set of incidences between $\mathcal{P}$ and Γ is defined as

$$I(\mathcal{P}, \Gamma) := \{(p, \gamma) \in \mathcal{P} \times \Gamma : p \in \gamma\}.$$

The incidence graph of Γ and $\mathcal{P}$, denoted $G_{\mathcal{P},\Gamma}$, is a bipartite graph $G_{\mathcal{P},\Gamma} = (V_\mathcal{P} \cup V_\Gamma, E)$, where the vertices $V_\mathcal{P}$ corresponding to points in $\mathcal{P}$ and the vertices V_Γ correspond to elements of Γ, and $(v_p, v_\gamma) \in E$ if $p \in \gamma$. Let $K_{s,t}$ denote the complete bipartite graph with s vertices on one side and t vertices on the other.

Theorem

Let $\vec{q}$ be an algebraically independent Pfaffian chain of order r and chain-degree α. Let $\mathcal{P} \subseteq \mathbb{R}^2$ be a set of points, and let $\Gamma \subseteq \mathbb{R}^2$ be a set of distinct irreducible Pfaffian curves such that each $\gamma \in \Gamma$ is defined by at most m Pfaffian functions defined w.r.t. $\vec{q}$ of degree at most β. Suppose that $G_{\mathcal{P},\Gamma}$ does not contain any copy of $K_{s,t}$. Then, for any $\varepsilon > 0$, there exist constants $C_1 = C_1(\varepsilon, m, \alpha, \beta, r, s, t)$ and $C_2 = C_2(\varepsilon, m, \alpha, \beta, r, s, t)$ such that

$$|I(\mathcal{P}, \Gamma)| \leq C_1\left(|\mathcal{P}|^{\frac{s(r+1)}{s(r+2)-1}+\varepsilon}|\Gamma|^{\frac{(s-1)(r+2)}{s(r+2)-1}}\right) + C_2\left(|\mathcal{P}| + |\Gamma|\right).$$

Application: Pfaffian Joints

Let $m, n \geq 2$. For $i \in [m]$, suppose Γ_i is a set of k_i-dimensional semi-Pfaffian sets in $\mathbb{R}^n$ such that $\sum_{i=1}^m k_i = n$. A point $x \in \mathbb{R}^n$ is a joint if

1. for each $i \in [m]$, there exists $\gamma_i \in \Gamma_i$ such that x is a smooth point of γ_i, and
2. the tangent spaces of γ_i at x for all i span $\mathbb{R}^n$.

Theorem

Let $\vec{q}$ be an algebraically independent Pfaffian chain of order r and chain-degree α. Suppose there are $n \geq 3$ sets of distinct irreducible Pfaffian curves $\{\Gamma_i\}_{i \in [n]}$ in $\mathbb{R}^n$ such that for all $i \in [n]$, each $\gamma_i \in \Gamma_i$ is defined by at most m Pfaffian functions defined w.r.t $\vec{q}$ of degree at most β. Then for any $\varepsilon > 0$, there exists a constant $C = C(\varepsilon, n, m, \alpha, \beta, r)$ such that the number of joints is bounded by

$$C \cdot \min_{j \in [n]}\left\{|\Gamma_j|^\varepsilon\right\} \cdot \prod_{i=1}^{n} |\Gamma_i|^{\max\left\{\frac{n+r}{n(n-1)}, \frac{2}{n+1}\right\}}.$$

Future Work

- Prove partitioning theorems for other classes of o-minimal structures
- Requires Bézout type theorems in different o-minimal structures
- Apply these techniques to other discrete geometry problems involving Pfaffian sets

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