Topology of random sets in semi-algebraic and o-minimal geometry with a view toward applications

Abhiram Natarajan

Advisors: Prof. Saugata Basu, Prof. Elena Grigorescu
Collaborators: Prof. Antonio Lerario (SISSA, Trieste), Prof. Joshua Grochow (U. Colorado, Boulder)

Outline

Superbvisor - Prof. Saugata Basu

contributions $= \aleph_0$

Superbvisor - Prof. Elena Grigorescu

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Collaborator - Prof. Antonio Lerario

great teacher, always teeming with ideas, very patient

Collaborator - Prof. Joshua Grochow

his niceness >> his supersonic brilliance = ∞

Others

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- Pavi and rest of my family
- Friends - Akash, Ashwin, Asish, Ganapathy, GV, Kaki, Kartik, Kaushal, Mayank, Negin, Omran, Onkar, Pavani, Rahul, Rohit, Sandeep, Shraddha, Sridhar, Vikhyat, Vikram, Vinit, Vivek, Warren

Outline

Real Algebraic Geometry

- Algebraic Set: The locus of common zeros of $\{P_1, \ldots, P_s\}$, $P_i \in \mathbb{R}[X_1, \ldots, X_n]$, i.e.

$$Z(P_1, \ldots, P_s) := \{x \in \mathbb{R}^n \mid P_1(x) = \ldots = P_s(x) = 0\}$$

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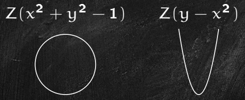

- **Semialgebraic set:** A set $S \subseteq \mathbb{R}^n$ that is a finite Boolean combination of sets of the form

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$\{-(x^2 + y^2 - 1) \geqslant 0\}$ $\quad$ $\{y \geqslant x\} \wedge \{x \geqslant y\}$ $\quad$ $\{x^2 + y^2 \leqslant 2\} \wedge (\{y - x \geqslant 4\} \vee \neg\{x - y \leqslant 4\})$

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Question

What is the average-case, and what does it even mean?

Distribution on Space of Polynomials

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- This is a natural measure

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- No points or directions are preferred in projective space

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  - Briquel and Bürgisser [2018] show that with standard Gaussian coefficients, $\mathbb{E}\,[\text{real zeros of } F] = O(mk^2t)$

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  - $b_2(X) =$ #two-dimensional *voids* or *cavities*, etc.

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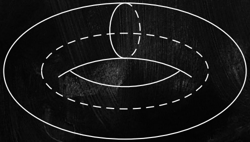	1	2	1	0

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- They offer a measure of complexity – e.g. height of algebraic computation tree for membership in semialgebraic set is lower bounded in terms of the Betti numbers (Yao 1997)
- In applications in incidence geometry, computational geometry, etc., especially for polynomial partitioning, bounds on Betti numbers of semi-algebraic sets are very important

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- Knowledge of the Betti numbers of arrangements, has been used for understanding "combinatorial complexity" (Basu 2002)

Previous work on Arrangements

- Sum of Betti nos. (Oleinik-Petrovski (1949), Thom (1965), Milnor (1964)) - $P_1, \ldots, P_s \in \mathbb{R}[X_1, \ldots, X_n]$, max degree d, then

$$\sum_{j \geqslant 0} b_j \left(\bigcup_{i=1}^{s} Z(P_i) \right) = O(s^n d^n)$$

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Question

What are the expected Betti numbers of an arrangement of random polynomials?

Expected Topology of Random Arrangements

Theorem (Basu-Lerario-N 2019b)

Let $P_1, \ldots, P_s \in \mathbb{R}[X_0, \ldots, X_n]$ *be homogeneous Kostlan forms, each of degree at most* d. *Let* $\Gamma_i \subset \mathbb{RP}^n$ *be the zero set of* P_i, *and define* $\Gamma = \bigcup_{i=1}^s \Gamma_i$. *Then*

$$\mathbb{E}\left[b_0(\mathbb{RP}^n \setminus \Gamma)\right] = 2s^n d^{n/2} + O\left(s^{n-1} d^{(n-1)/2}\right).$$

Also, for $0 < i \leqslant n-1$

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Interpretation

Worst-case bound on b_0 *is* $\binom{s}{n} O(d^n)$, *while expectation is equal to* $2s^n d^{n/2}$.

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$$\sum_{k \geqslant 0} b_k(S) \leqslant n^{O(s)}$$

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What is the expected Betti number of a union of random quadrics?

b_0 of Quadrics' Arrangement

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Interpretation

Our general theorem suggests $\mathbb{E}\left[b_0(\Gamma)\right] = O(s)$*. For quadrics, we prove* $\mathbb{E}\left[b_0(\Gamma)\right] = o(s)$*.*

Quadrics Arrangement – Proof

- Let $\mathrm{Sym}(n+1,\mathbb{R})$ be the vector space of $(n+1)\times(n+1)$ real symmetric matrices; we have
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- $\mathbb{RP}^N = \mathbb{P}(\mathrm{Sym}(n+1,\mathbb{R}))$ - projectivization of the space of symmetric matrices (here $N = \binom{n+2}{2} - 1$)

- Turns out sampling a Kostlan quadric is equivalent to sampling uniformly at random from S^N

Characterization of 'Quadrics' Intersection

Theorem (Calabi 1964)

For $n \geqslant 1$ *let* $q_1, q_2 \in \mathbb{R}[x_0, \ldots, x_n]_{(2)}$ *and denote by* $\Gamma_1, \Gamma_2 \subset \mathbb{RP}^n$ *their (possibly empty) zero sets. Let* $\mathcal{P}_n \subseteq S^N$ *denote the set of positive quadratic forms. Let* $\ell \subset S^N$ *be the projective line* $\ell = \{[\lambda_1 q_1 + \lambda_2 q_2]\}_{\lambda_i \in \mathbb{RP}^1}$ *(a pencil of quadrics). Then:*

$$\Gamma_1 \cap \Gamma_2 \neq \emptyset \iff \ell \cap \mathcal{P}_n = \emptyset.$$

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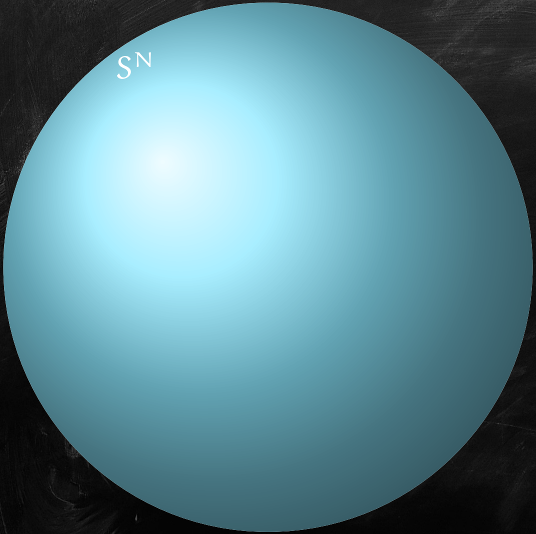
- *Join points iff the great circle joining points does not pass through $\mathcal{P}_n$*

Illustration of 'Obstacle' Random Graph



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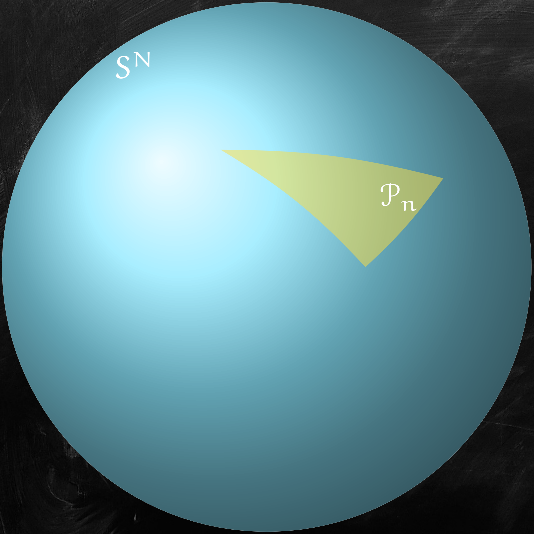

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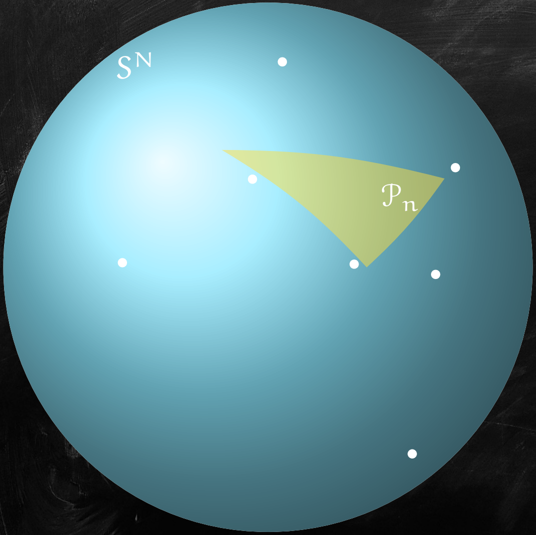

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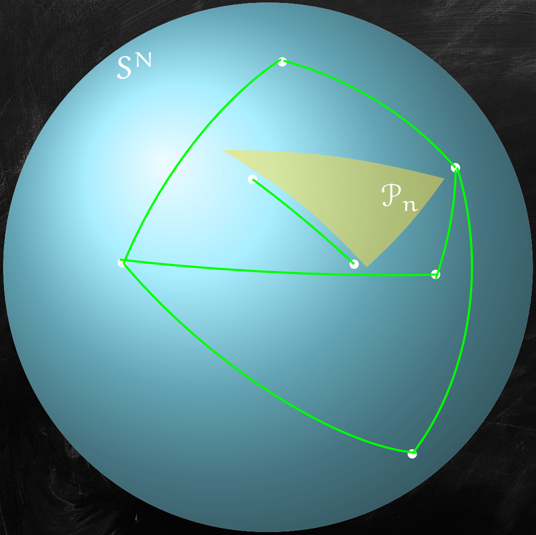

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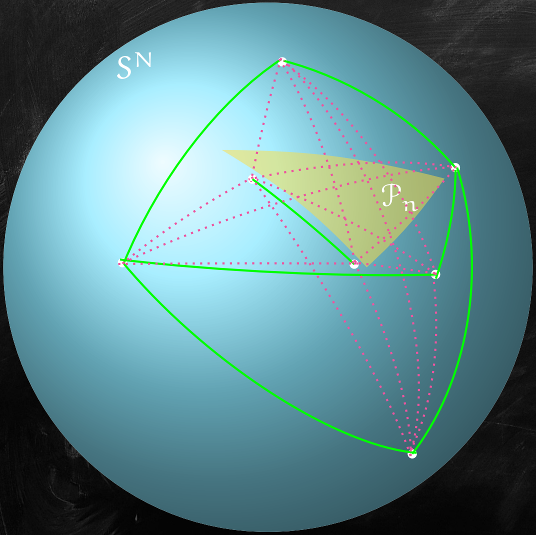

Obstacle Random Graph – Properties

- Good cone: for $q \in S^N$

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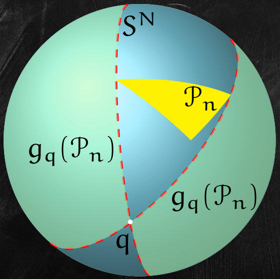

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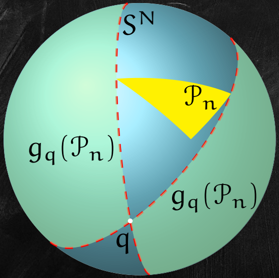

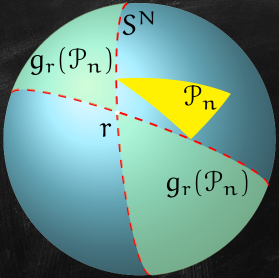

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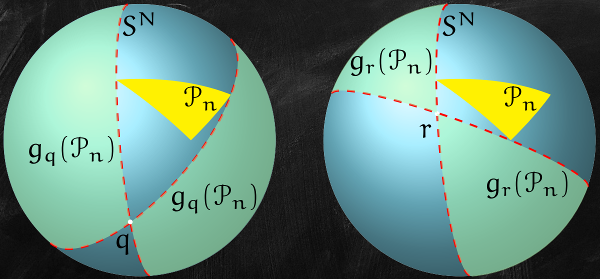

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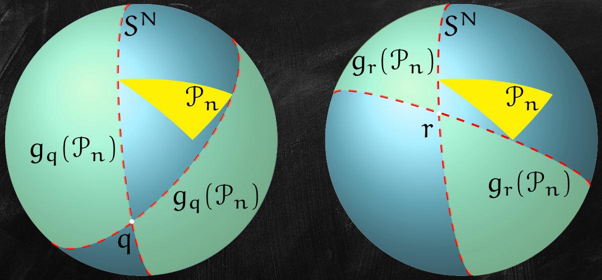

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Question

What is the average number of connected components in the above random graph?

Average Connected Components

Theorem (Basu-Lerario-N 2019b)

The expected number of connected component of $\mathcal{G}(N, \mathcal{P}_n, s)$ *satisfies:*

$$\lim_{s \to \infty} \frac{\mathbb{E}\left[b_0(\mathcal{G}(N, \mathcal{P}_n, s))\right]}{s} \leqslant \frac{\mathrm{vol}\left(\mathcal{P}_n\right)}{\mathrm{vol}\left(\mathbb{RP}^N\right)}.$$

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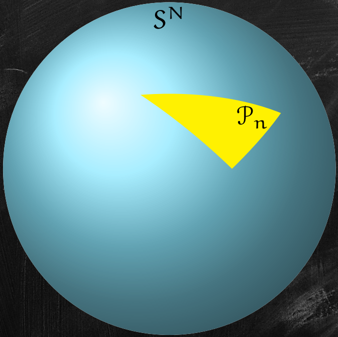
Interpretation

Considering $\frac{\text{vol}(\mathcal{P}_n)}{\text{vol}(\mathbb{RP}^N)}$ *to be fixed, we have that the expected number of connected components is* $o(s)$.

Average Connected Components – Proof



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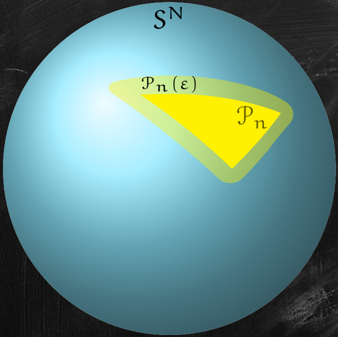

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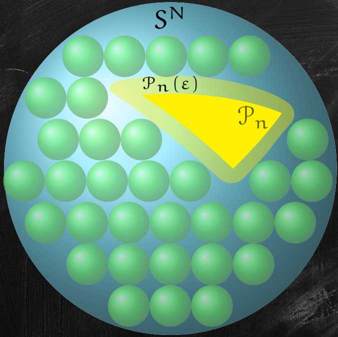

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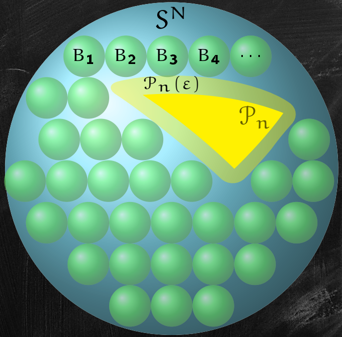

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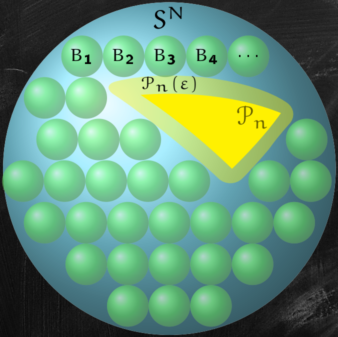

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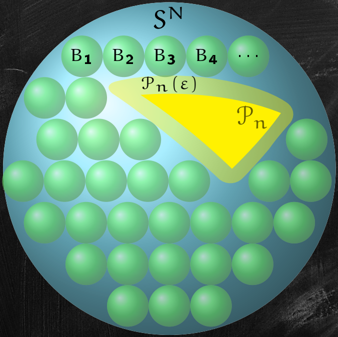

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- Using coupon-collector type argument, bound number of samples required to collect all B_i. ■

Future Work

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- We prove a Ramsey theoretic result - we prove large cliques will exist in the graph w.h.p.

Future Work

- A sign condition on $P_1, \ldots, P_s$ is the locus of e.g. $P_1(x) < 0 \wedge P_2(x) > 0 \wedge \ldots \wedge P_s(x) < 0$

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Future Questions:

- What is the probability of a sign condition on $P_1, \ldots, P_s$ to be realizable?
- What are the expected Betti numbers of sign conditions?

Outline

Generalize Semi-algebraic Geometry

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Question

"...investigate classes of sets with the tame topological properties of semialgebraic sets..." - Grothendieck (Esquisse d'un Programme, 1997)

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☆ Elements of $\mathcal{S}_1$ are precisely finite unions of points and intervals

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- Definable sets have a 'tame topology'

Betti Numbers of Definable Sets

- Real Analogue of Bezout theorem (Barone-Basu 2016): Given $\deg(Q) \ll \deg(P)$, $\dim(Z(Q)) = k$, then
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Question

Given a definable hypersurface γ, and a degree D polynomial $P \in \mathbb{R}[X_1, \ldots, X_n]$, bound $b_k(\gamma \cap Z(P))$.

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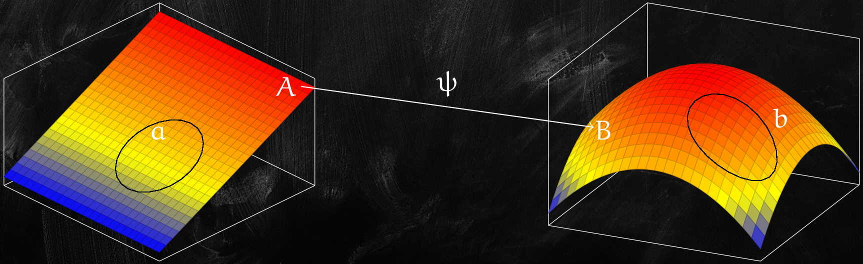

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How 'common' is the pathological case?

Average Topology of Definable Hypersurfaces on Algebraic Sets

Theorem (Basu-Lerario-N (2019a))

Let $\Gamma \subset \mathbb{RP}^n$ *be a regular, compact semi-analytic hypersurface, and let* p *be a random Kostlan polynomial of degree* D. *Then there exists a constant* c_Γ *such that for every* $0 \leqslant k \leqslant n-2$, *for every* $t > 0$

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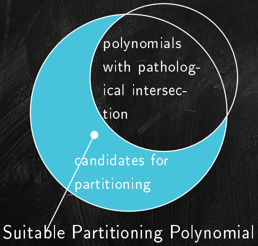

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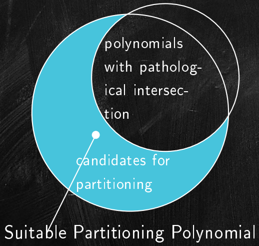

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