Methods in Algebraic Geometry towards Data Science Applications

Abhiram Natarajan

University of Warwick, UK

Outline

Introduction

Gröbner Bases

Applications of Gröbner Bases

Conclusion

Today's Presentation

- I do research in algebraic geometry/topology

Today's Presentation

- I do research in algebraic geometry/topology
- Algebraic geometry deals with non-linear objects, i.e. polynomials

Today's Presentation

- I do research in algebraic geometry/topology
- Algebraic geometry deals with non-linear objects, i.e. polynomials
- Non-linear methods are increasingly being used in data science/mining, e.g., finding *missing links* in palaeontology

Today's Presentation

- I do research in algebraic geometry/topology
- Algebraic geometry deals with non-linear objects, i.e. polynomials
- Non-linear methods are increasingly being used in data science/mining, e.g., finding *missing links* in palaeontology

Question

What is algebraic geometry? What do we mean by non-linear?

Algebraic Geometry

- Algebraic Geometry studies zeros of polymonials:
  given polynomial $f(X)$, find $Z(f) := \{X \text{ such that } f(X) = 0\}$

Algebraic Geometry

- Algebraic Geometry studies zeros of polymonials:
  given polynomial $f(X)$, find $Z(f) := \{X \text{ such that } f(X) = 0\}$
- Examples:

Algebraic Geometry

- Algebraic Geometry studies zeros of polymonials:
  given polynomial $f(X)$, find $Z(f) := \{X \text{ such that } f(X) = 0\}$
- Examples:

Polynomial	Zeros
$f(x) = x - 7$	

Algebraic Geometry

- Algebraic Geometry studies zeros of polymonials:
  given polynomial $f(X)$, find $Z(f) := \{X \text{ such that } f(X) = 0\}$
- Examples:

Polynomial	Zeros
$f(x) = x - 7$	$Z(f) = \{7\}$

Algebraic Geometry

- Algebraic Geometry studies zeros of polymonials:
  given polynomial $f(X)$, find $Z(f) := \{X \text{ such that } f(X) = 0\}$

- Examples:

Polynomial	Zeros
$f(x) = x - 7$	$Z(f) = \{7\}$
$f(x) = x^2 - 3x + 2$	

Algebraic Geometry

- Algebraic Geometry studies zeros of polymonials:
  given polynomial $f(X)$, find $Z(f) := \{X \text{ such that } f(X) = 0\}$

- Examples:

Polynomial	Zeros
$f(x) = x - 7$	$Z(f) = \{7\}$
$f(x) = x^2 - 3x + 2$ $= (x-1)(x-2)$	

Algebraic Geometry

- Algebraic Geometry studies zeros of polymonials:
  given polynomial $f(X)$, find $Z(f) := \{X \text{ such that } f(X) = 0\}$
- Examples:

Polynomial	Zeros
$f(x) = x - 7$	$Z(f) = \{7\}$
$f(x) = x^2 - 3x + 2$ $= (x-1)(x-2)$	$Z(f) = \{1, 2\}$

Algebraic Geometry

- Algebraic Geometry studies zeros of polymonials:
  given polynomial $f(X)$, find $Z(f) := \{X \text{ such that } f(X) = 0\}$
- Examples:

Polynomial	Zeros
$f(x) = x - 7$	$Z(f) = \{7\}$
$f(x) = x^2 - 3x + 2$ $= (x-1)(x-2)$	$Z(f) = \{1, 2\}$
$f(x) = ax^2 + bx + c$	

Algebraic Geometry

- Algebraic Geometry studies zeros of polymonials:
  given polynomial $f(X)$, find $Z(f) := \{X \text{ such that } f(X) = 0\}$

- Examples:

Polynomial	Zeros
$f(x) = x - 7$	$Z(f) = \{7\}$
$f(x) = x^2 - 3x + 2$ $= (x-1)(x-2)$	$Z(f) = \{1, 2\}$
$f(x) = ax^2 + bx + c$	$Z(f) = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Algebraic Geometry

- More examples:

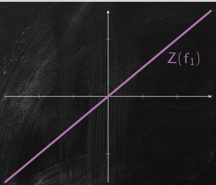
Algebraic Geometry

- More examples:

Polynomial	Zeros
$f_1(x,y) = y - x$	

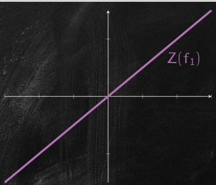
Algebraic Geometry

- More examples:

Polynomial	Zeros
$f_1(x, y) = y - x$	

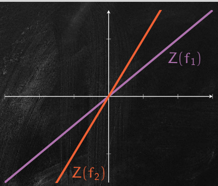
Algebraic Geometry

- More examples:

Polynomial	Zeros
$f_1(x,y) = y - x$ $f_2(x,y) = y - 2x$	

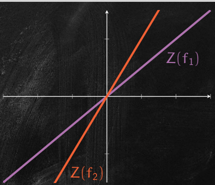
Algebraic Geometry

- More examples:

Polynomial	Zeros
$f_1(x,y) = y - x$ $f_2(x,y) = y - 2x$	

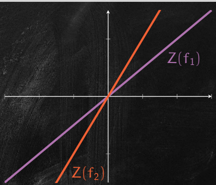
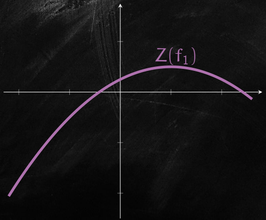
Algebraic Geometry

- More examples:

Polynomial	Zeros
$f_1(x,y) = y - x$ $f_2(x,y) = y - 2x$	
$f_1(x,y) = y + x^2 - 2x - 1$	

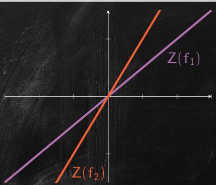
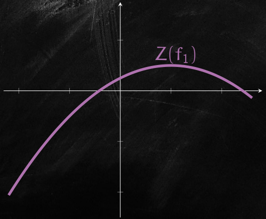
Algebraic Geometry

- More examples:

Polynomial	Zeros
$f_1(x,y) = y - x$ $f_2(x,y) = y - 2x$	
$f_1(x,y) = y + x^2 - 2x - 1$	

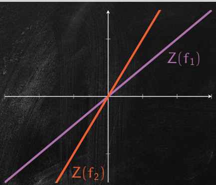
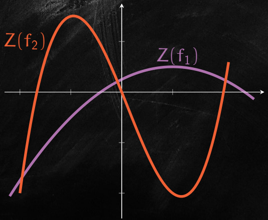
Algebraic Geometry

- More examples:

Polynomial	Zeros
$f_1(x,y) = y - x$ $f_2(x,y) = y - 2x$	
$f_1(x,y) = y + x^2 - 2x - 1$ $f_2(x,y) = y - 3x^3 + x^2 + 10x$	

Algebraic Geometry

- More examples:

Polynomial	Zeros
$f_1(x,y) = y - x$ $f_2(x,y) = y - 2x$	
$f_1(x,y) = y + x^2 - 2x - 1$ $f_2(x,y) = y - 3x^3 + x^2 + 10x$	

Algebraic Geometry

- More examples:

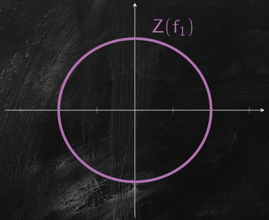
Algebraic Geometry

- More examples:

Polynomial	Zeros
$f_1(x, y) = x^2 + y^2 - 1$	

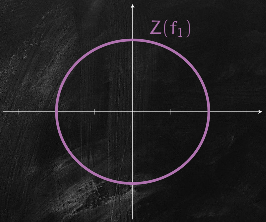
Algebraic Geometry

- More examples:

Polynomial	Zeros
$f_1(x,y) = x^2 + y^2 - 1$	

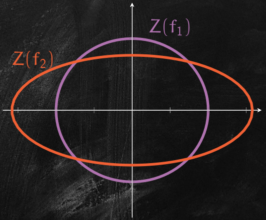
Algebraic Geometry

- More examples:

Polynomial	Zeros
$f_1(x,y) = x^2 + y^2 - 1$ $f_2(x,y) = 17x^2 + 4y^2 - 10$	

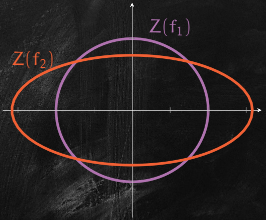
Algebraic Geometry

- More examples:

Polynomial	Zeros
$f_1(x, y) = x^2 + y^2 - 1$ $f_2(x, y) = 17x^2 + 4y^2 - 10$	

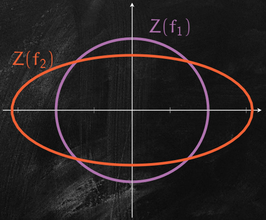
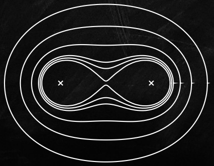
Algebraic Geometry

- More examples:

Polynomial	Zeros
$f_1(x, y) = x^2 + y^2 - 1$ $f_2(x, y) = 17x^2 + 4y^2 - 10$	
Cassini Ovals	

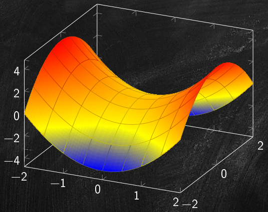
Algebraic Geometry

- More examples:

Polynomial	Zeros
$f_1(x, y) = x^2 + y^2 - 1$ $f_2(x, y) = 17x^2 + 4y^2 - 10$	
Cassini Ovals	

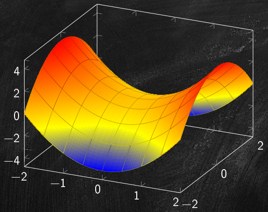
Algebraic Geometry

- Alg. Geom. deals with all kinds of n-dimensional surfaces!



Algebraic Geometry

- Alg. Geom. deals with all kinds of n-dimensional surfaces!



Question

Given polynomials $f_1(X), \ldots, f_m(X)$, *can you solve the system*

$$\Phi := \begin{cases} f_1(X) = 0 \\ \vdots \\ f_m(X) = 0 \end{cases} \quad ?$$

'Non-linear' Algebra

- $f(X)$ is linear if $f(\alpha \cdot X) = \alpha \cdot f(X)$

'Non-linear' Algebra

- $f(X)$ is linear if $f(\alpha \cdot X) = \alpha \cdot f(X)$

  e.g. $f(x) = 7x, \qquad g(x, y) = 15x + 7y$

'Non-linear' Algebra

- $f(X)$ is linear if $f(\alpha \cdot X) = \alpha \cdot f(X)$

  e.g. $f(x) = 7x, \qquad g(x, y) = 15x + 7y$

- Solving $\Phi = \begin{cases} f_1(X)=0 \\ \vdots \\ f_m(X)=0 \end{cases}$ when the f_i are linear is done using Gaussian elimination

'Non-linear' Algebra

- $f(X)$ is linear if $f(\alpha \cdot X) = \alpha \cdot f(X)$

  e.g. $f(x) = 7x, \qquad g(x, y) = 15x + 7y$

- Solving $\Phi = \begin{cases} f_1(X)=0 \\ \vdots \\ f_m(X)=0 \end{cases}$ when the f_i are linear is done using Gaussian elimination

- Today we are going to learn about Gröbner Bases, which allows us to solve Φ for non-linear f_i, i.e. polynomials

Outline

Introduction

Gröbner Bases

Applications of Gröbner Bases

Conclusion

Solving System of Linear Equations

Two equations, two variables:

Solving System of Linear Equations

Two equations, two variables:

$$\begin{aligned} 3x &+ & y &= -1 \\ 7x &+ & 11y &= 15 \end{aligned}$$

Solving System of Linear Equations

Two equations, two variables:

$$\begin{aligned} 3x &+ y &= -1 \\ 7x &+ 11y &= 15 \end{aligned}$$

Solving System of Linear Equations

Two equations, two variables:

$$7 \times \left(\begin{array}{cccc} 3x & + & y & = & -1 \end{array} \right)$$

$$7x + 11y = 15$$

Solving System of Linear Equations

Two equations, two variables:

$$7 \times \left(3x + y = -1 \right)$$

$$3 \times \left(7x + 11y = 15 \right)$$

Solving System of Linear Equations

Two equations, two variables:

$$
\begin{array}{rl}
 & 7 \times \left(\; 3x + y = -1 \; \right) \\
- & \\
 & 3 \times \left(\; 7x + 11y = 15 \; \right) \\
\hline
\end{array}
$$

Solving System of Linear Equations

Two equations, two variables:

$$
\begin{array}{rl}
 & 7 \times \left(\begin{array}{ccccc} 3x & + & y & = & -1 \end{array} \right) \\
- & \\
 & 3 \times \left(\begin{array}{ccccc} 7x & + & 11y & = & 15 \end{array} \right) \\
\hline
 & \begin{array}{ccccc} 0 & + & -26y & = & -52 \end{array} \\
\hline
\end{array}
$$

Solving System of Linear Equations

Two equations, two variables:

$$
\begin{array}{rl}
 & 7 \times \left(\quad 3x \quad + \quad y \quad = \quad -1 \quad \right) \\
- & \\
 & 3 \times \left(\quad 7x \quad + \quad 11y \quad = \quad 15 \quad \right) \\
\hline
 & \quad 0 \quad + \quad -26y \quad = \quad -52 \\
\hline
\end{array}
$$

substitute $y = 2$ into either equation to get $x = -1$

Gaussian Elimination

- In general, given a system of n linear equations in n-variables, eliminate leading term for every pair of linear equations

Gaussian Elimination

- In general, given a system of n linear equations in n-variables, eliminate leading term for every pair of linear equations
  - e.g. Given $f = 4x + 7y - 23z$, $g = 7x - 23y$ eliminate x term by computing $7f - 4g$

Gaussian Elimination

- In general, given a system of n linear equations in n-variables, eliminate leading term for every pair of linear equations
  - e.g. Given $f = 4x + 7y - 23z$, $g = 7x - 23y$ eliminate x term by computing $7f - 4g$
  - f is multiplied by the leading co-efficient of g, i.e. $7f$

Gaussian Elimination

- In general, given a system of n linear equations in n-variables, eliminate leading term for every pair of linear equations
  - e.g. Given $f = 4x + 7y - 23z$, $g = 7x - 23y$ eliminate x term by computing $7f - 4g$
  - f is multiplied by the leading co-efficient of g, i.e. $7f$
  - g is multiplied by the leading co-efficient of f, i.e. $4g$

Gaussian Elimination

- In general, given a system of n linear equations in n-variables, eliminate leading term for every pair of linear equations
  - e.g. Given $f = 4x + 7y - 23z$, $g = 7x - 23y$ eliminate x term by computing $7f - 4g$
  - f is multiplied by the leading co-efficient of g, i.e. $7f$
  - g is multiplied by the leading co-efficient of f, i.e. $4g$
- Continue until you reach an equation with just one variable

Gaussian Elimination

- In general, given a system of n linear equations in n-variables, eliminate leading term for every pair of linear equations
  - e.g. Given $f = 4x + 7y - 23z$, $g = 7x - 23y$ eliminate x term by computing $7f - 4g$
  - f is multiplied by the leading co-efficient of g, i.e. $7f$
  - g is multiplied by the leading co-efficient of f, i.e. $4g$
- Continue until you reach an equation with just one variable
- Solve system with back substitution

Gaussian Elimination

- In general, given a system of n linear equations in n-variables, eliminate leading term for every pair of linear equations
  - e.g. Given $f = 4x + 7y - 23z$, $g = 7x - 23y$ eliminate x term by computing $7f - 4g$
  - f is multiplied by the leading co-efficient of g, i.e. $7f$
  - g is multiplied by the leading co-efficient of f, i.e. $4g$
- Continue until you reach an equation with just one variable
- Solve system with back substitution

Question

Can this idea be applied to a system of polynomials?

Solving System of Non-linear Equations

Two polynomials:

$$
\begin{aligned}
x^2y^3 - 4 &= 0 \\
x^3y^2 - 2 &= 0
\end{aligned}
$$

Solving System of Non-linear Equations

Two polynomials:

$$x^2y^3 - 4 = 0$$

$$x^3y^2 - 2 = 0$$

Solving System of Non-linear Equations

Two polynomials:

$$x \times \left(\begin{array}{cccc} x^2y^3 & - & 4 & = & 0 \\ \\ x^3y^2 & - & 2 & = & 0 \end{array} \right.$$

Solving System of Non-linear Equations

Two polynomials:

$$x \times \left(x^2y^3 - 4 = 0 \right)$$

$$y \times \left(x^3y^2 - 2 = 0 \right)$$

Solving System of Non-linear Equations

Two polynomials:

$$
\begin{array}{rl}
 & x \times \left(\quad x^2y^3 \quad - \quad 4 \quad = \quad 0 \quad \right) \\
- & \\
 & y \times \left(\quad x^3y^2 \quad - \quad 2 \quad = \quad 0 \quad \right) \\
\hline
\end{array}
$$

Solving System of Non-linear Equations

Two polynomials:

$$
\begin{array}{rl}
 & x \times \left(\; x^2y^3 \;-\; 4 \;=\; 0 \; \right) \\
- & \\
 & y \times \left(\; x^3y^2 \;-\; 2 \;=\; 0 \; \right) \\
\hline
 & \quad 0 \;+\; 4x - 2y \;=\; 0
\end{array}
$$

Solving System of Non-linear Equations

Two polynomials:

$$
\begin{array}{rl}
 & x \times \left(x^2y^3 - 4 = 0 \right) \\
- & y \times \left(x^3y^2 - 2 = 0 \right) \\
\hline
 & 0 + 4x - 2y = 0 \\
\Longrightarrow & y = 2x \\
\hline
\end{array}
$$

Solving System of Non-linear Equations

Two polynomials:

$$
\begin{array}{rlcccl}
 & x \times \big(& x^2y^3 & - & 4 & = 0 \big) \\
- & & & & & \\
 & y \times \big(& x^3y^2 & - & 2 & = 0 \big) \\
\hline
 & & 0 & + & 4x - 2y & = 0 \\
\Longrightarrow & & & & y & = 2x \\
\hline
\end{array}
$$

1. put $y = 2x$ in first equation to get $4x^3 = 4$, thus $x = 1$

Solving System of Non-linear Equations

Two polynomials:

$$
\begin{array}{rrcccc}
 & x \times \big(& x^2y^3 & - & 4 & = & 0 \big) \\
- & & & & & & \\
 & y \times \big(& x^3y^2 & - & 2 & = & 0 \big) \\
\hline
 & & 0 & + & 4x - 2y & = & 0 \\
\implies & & & & y & = & 2x \\
\hline
\end{array}
$$

1. put $y = 2x$ in first equation to get $4x^3 = 4$, thus $x = 1$
2. put $x = 1$ in '$y = 2x$' to get $y = 2$

Generalized Gaussian Elimination

- Modified Gaussian elimination works in the non-linear case!

Generalized Gaussian Elimination

- Modified Gaussian elimination works in the non-linear case!
- For f, g, polynomial obtained after eliminating leading terms of f and g is called S-polynomial of f and g, denoted $S(f, g)$

Generalized Gaussian Elimination

- Modified Gaussian elimination works in the non-linear case!
- For f, g, polynomial obtained after eliminating leading terms of f and g is called S-polynomial of f and g, denoted $S(f, g)$
  - $f = x^2z - 5x^2y + 12x + 23$, $g = xy + 2$

Generalized Gaussian Elimination

- Modified Gaussian elimination works in the non-linear case!
- For f, g, polynomial obtained after eliminating leading terms of f and g is called S-polynomial of f and g, denoted $S(f, g)$
  - $f = x^2z - 5x^2y + 12x + 23$, $g = xy + 2$
  - Leading term of f is x^2z, leading term of g is xy

Generalized Gaussian Elimination

- Modified Gaussian elimination works in the non-linear case!
- For f, g, polynomial obtained after eliminating leading terms of f and g is called S-polynomial of f and g, denoted $S(f, g)$
  - $f = x^2z - 5x^2y + 12x + 23$, $g = xy + 2$
  - Leading term of f is x^2z, leading term of g is xy
  - $S(f, g) = yf - xzg$, eliminates leading terms of both f and g

Generalized Gaussian Elimination

- Modified Gaussian elimination works in the non-linear case!
- For f, g, polynomial obtained after eliminating leading terms of f and g is called S-polynomial of f and g, denoted $S(f, g)$
  - $f = x^2z - 5x^2y + 12x + 23$, $g = xy + 2$
  - Leading term of f is x^2z, leading term of g is xy
  - $S(f, g) = yf - xzg$, eliminates leading terms of both f and g
- Compute S-polynomials of every pair of polynomials until system can be solved

Generalized Gaussian Elimination

- Modified Gaussian elimination works in the non-linear case!
- For f, g, polynomial obtained after eliminating leading terms of f and g is called S-polynomial of f and g, denoted $S(f, g)$
  - $f = x^2z - 5x^2y + 12x + 23, \ g = xy + 2$
  - Leading term of f is x^2z, leading term of g is xy
  - $S(f, g) = yf - xzg$, eliminates leading terms of both f and g
- Compute S-polynomials of every pair of polynomials until system can be solved
- The set of all S-polynomials is called a Gröbner basis

Buchberger's Algorithm for Gröbner Basis

Algorithm 1 Compute Gröbner basis of polynomial system

Buchberger's Algorithm for Gröbner Basis

Algorithm 1 Compute Gröbner basis of polynomial system

Require: Set of polynomials $S = \{f_1, \ldots, f_m\}$

```
function Gröbner-Basis(S)
    do
```

Buchberger's Algorithm for Gröbner Basis

Algorithm 1 Compute Gröbner basis of polynomial system

Require: Set of polynomials $S = \{f_1, \ldots, f_m\}$

```
function Gröbner-Basis(S)
    do
        S' ← ∅
```

Buchberger's Algorithm for Gröbner Basis

Algorithm 1 Compute Gröbner basis of polynomial system

Require: Set of polynomials $S = \{f_1, \ldots, f_m\}$

function Gröbner-Basis(S)
    do
        $S' \leftarrow \emptyset$
        for each $f, g \in S$ **do**

Buchberger's Algorithm for Gröbner Basis

Algorithm 1 Compute Gröbner basis of polynomial system

Require: Set of polynomials $S = \{f_1, \ldots, f_m\}$

```
function Gröbner-Basis(S)
    do
        S′ ← ∅
        for each f, g ∈ S do
            h ← S(f, g)
```

Buchberger's Algorithm for Gröbner Basis

Algorithm 1 Compute Gröbner basis of polynomial system

Require: Set of polynomials $S = \{f_1, \ldots, f_m\}$

```
function Gröbner-Basis(S)
    do
        S' ← ∅
        for each f, g ∈ S do
            h ← S(f, g)
            S' ← S' ∪ {h}
```

Buchberger's Algorithm for Gröbner Basis

Algorithm 1 Compute Gröbner basis of polynomial system

Require: Set of polynomials $S = \{f_1, \ldots, f_m\}$

function Gröbner-Basis(S)
  do
    $S' \leftarrow \emptyset$
    for each $f, g \in S$ **do**
      $h \leftarrow S(f, g)$
      $S' \leftarrow S' \cup \{h\}$
    $S \leftarrow S \cup S'$

Buchberger's Algorithm for Gröbner Basis

Algorithm 1 Compute Gröbner basis of polynomial system

Require: Set of polynomials $S = \{f_1, \ldots, f_m\}$

function Gröbner-Basis(S)
    do
        $S' \leftarrow \emptyset$
        for each $f, g \in S$ **do**
            $h \leftarrow S(f, g)$
            $S' \leftarrow S' \cup \{h\}$
        $S \leftarrow S \cup S'$
    while S is not solvable

Buchberger's Algorithm for Gröbner Basis

Algorithm 1 Compute Gröbner basis of polynomial system

Require: Set of polynomials $S = \{f_1, \ldots, f_m\}$

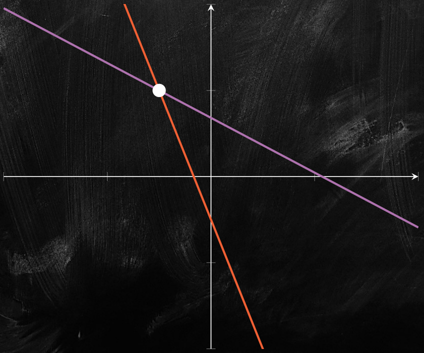
function Gröbner-Basis(S)
    do
        $S' \leftarrow \emptyset$
        for each $f, g \in S$ **do**
            $h \leftarrow S(f, g)$
            $S' \leftarrow S' \cup \{h\}$
        $S \leftarrow S \cup S'$
    while S is not solvable
    return S

Geometry of Gaussian Elimination

$$\begin{aligned} 3x &+ y &= -1 \\ 7x &+ 11y &= 15 \end{aligned}$$

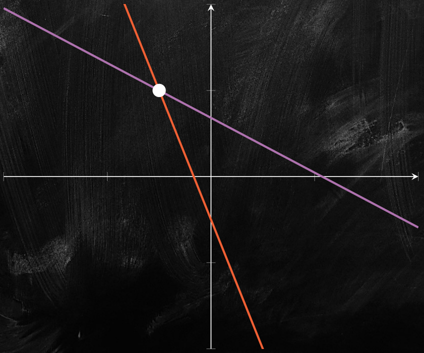
Geometry of Gaussian Elimination

$$\begin{aligned} 3x &+ y &= -1 \\ 7x &+ 11y &= 15 \end{aligned}$$



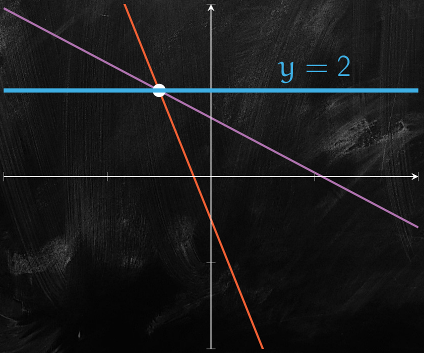
Geometry of Gaussian Elimination

$$
\begin{array}{rrcrcrcrl}
 & 7 \times (& 3x & + & y & = & -1 &) \\
- & 3 \times (& 7x & + & 11y & = & 15 &) \\
\hline
 & & 0 & + & -26y & = & -52 & \\
\hline
\end{array}
$$



Geometry of Gaussian Elimination

$$\begin{array}{rrcrcrcrl} & 7 \times (& 3x & + & y & = & -1 &) \\ - & 3 \times (& 7x & + & 11y & = & 15 &) \\ \hline & & 0 & + & -26y & = & -52 & \\ \hline \end{array}$$

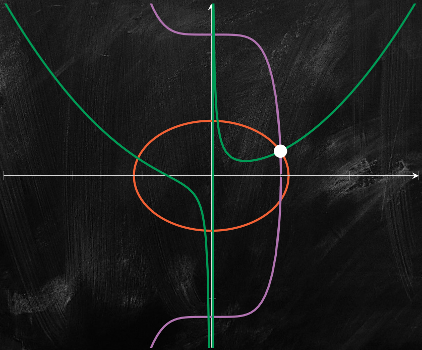


Geometry of Gröbner Basis

$x^2 + y^2 - 5 = 0 \quad x^5 + y^2 - 33 = 0 \quad x^3 - 5xy + 2 = 0$

Geometry of Gröbner Basis

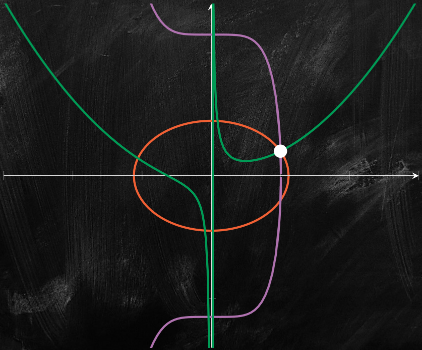
$x^2 + y^2 - 5 = 0 \quad x^5 + y^2 - 33 = 0 \quad x^3 - 5xy + 2 = 0$



Geometry of Gröbner Basis

$x^2 + y^2 - 5 = 0 \quad x^5 + y^2 - 33 = 0 \quad x^3 - 5xy + 2 = 0$

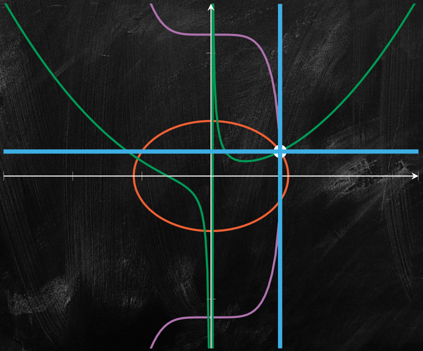
$\text{Gröbner-Basis} = \{y - 1, x - 2\}$



Geometry of Gröbner Basis

$x^2 + y^2 - 5 = 0 \quad x^5 + y^2 - 33 = 0 \quad x^3 - 5xy + 2 = 0$

Gröbner-Basis $= \{y - 1, x - 2\}$



Outline

Application – Polynomial System Solvability

- S is not solvable $\iff$ Gröbner-Basis(S) contains a constant polynomial

Application – Polynomial System Solvability

- S is not solvable $\iff$ Gröbner-Basis(S) contains a `constant polynomial`

Polynomial	Gröbner Basis	Zeros

Application – Polynomial System Solvability

- S is not solvable $\iff$ Gröbner-Basis(S) contains a constant polynomial

Polynomial	Gröbner Basis	Zeros
unsolvable system $f_1(x, y) = y - x + 2$ $f_2(x, y) = y - x - 1$		

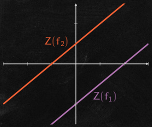
Application – Polynomial System Solvability

- S is not solvable $\iff$ Gröbner-Basis(S) contains a `constant polynomial`

Polynomial	Gröbner Basis	Zeros
unsolvable system $f_1(x, y) = y - x + 2$ $f_2(x, y) = y - x - 1$	`i2 : gens gb ideal(y - x + 2, y - x - 1)` `o2 = | 1 |`	

Application – Polynomial System Solvability

- S is not solvable $\iff$ Gröbner-Basis(S) contains a constant polynomial

Polynomial	Gröbner Basis	Zeros
unsolvable system $f_1(x, y) = y - x + 2$ $f_2(x, y) = y - x - 1$	`i2 : gens gb ideal(y - x + 2, y - x - 1)` `o2 = | 1 |`	

Application – Polynomial System Solvability

- S is not solvable $\iff$ Gröbner-Basis(S) contains a constant polynomial

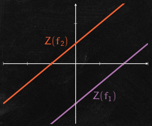
Polynomial | **Gröbner Basis** | **Zeros**

unsolvable system

$$f_1(x,y) = y - x + 2$$
$$f_2(x,y) = y - x - 1$$

```
i2 : gens gb ideal(y - x + 2, y - x - 1)

o2 = | 1 |
```

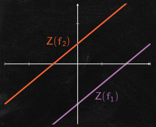


solvable system

$$f_1(x,y) = y - x + 2$$
$$f_2(x,y) = y + x$$

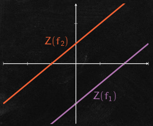
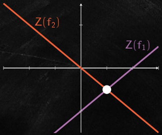
Application – Polynomial System Solvability

- S is not solvable $\iff$ Gröbner-Basis(S) contains a `constant polynomial`

Polynomial	Gröbner Basis	Zeros
unsolvable system $f_1(x,y) = y - x + 2$ $f_2(x,y) = y - x - 1$	`i2 : gens gb ideal(y - x + 2, y - x - 1)` `o2 = | 1 |`	
solvable system $f_1(x,y) = y - x + 2$ $f_2(x,y) = y + x$	`i2 : gens gb ideal(y - x + 2, y + x)` `o2 = | y+1 x-1 |`	

Application – Polynomial System Solvability

- S is not solvable $\iff$ Gröbner-Basis(S) contains a `constant polynomial`

Polynomial	Gröbner Basis	Zeros
unsolvable system $f_1(x,y) = y - x + 2$ $f_2(x,y) = y - x - 1$	`i2 : gens gb ideal(y - x + 2, y - x - 1)` `o2 = | 1 |`	
solvable system $f_1(x,y) = y - x + 2$ $f_2(x,y) = y + x$	`i2 : gens gb ideal(y - x + 2, y + x)` `o2 = | y+1 x-1 |`	

Application - Sudoku Solving

x_{11}	x_{12}	x_{13}	x_{14}
x_{21}	x_{22}	x_{23}	x_{24}
x_{31}	x_{32}	x_{33}	x_{34}
x_{41}	x_{42}	x_{43}	x_{44}

Application – Sudoku Solving

x_{11}	x_{12}	x_{13}	x_{14}
x_{21}	x_{22}	x_{23}	x_{24}
x_{31}	x_{32}	x_{33}	x_{34}
x_{41}	x_{42}	x_{43}	x_{44}

$$x_{11}+x_{12}+x_{13}+x_{14}=10$$
$$x_{21}+x_{22}+x_{23}+x_{24}=10$$
$$x_{31}+x_{32}+x_{33}+x_{34}=10$$
$$x_{41}+x_{42}+x_{43}+x_{44}=10$$

Application – Sudoku Solving

x_{11}	x_{12}	x_{13}	x_{14}	$x_{11}+x_{12}+x_{13}+x_{14}=10$
x_{21}	x_{22}	x_{23}	x_{24}	$x_{21}+x_{22}+x_{23}+x_{24}=10$
x_{31}	x_{32}	x_{33}	x_{34}	$x_{31}+x_{32}+x_{33}+x_{34}=10$
x_{41}	x_{42}	x_{43}	x_{44}	$x_{41}+x_{42}+x_{43}+x_{44}=10$

$$x_{11}+x_{21}+x_{31}+x_{41}=10$$
$$x_{12}+x_{22}+x_{32}+x_{42}=10$$
$$x_{13}+x_{23}+x_{33}+x_{43}=10$$
$$x_{14}+x_{24}+x_{34}+x_{44}=10$$

Application – Sudoku Solving

x_{11}	x_{12}	x_{13}	x_{14}
x_{21}	x_{22}	x_{23}	x_{24}
x_{31}	x_{32}	x_{33}	x_{34}
x_{41}	x_{42}	x_{43}	x_{44}

$$x_{11}+x_{12}+x_{13}+x_{14}=10$$
$$x_{21}+x_{22}+x_{23}+x_{24}=10$$
$$x_{31}+x_{32}+x_{33}+x_{34}=10$$
$$x_{41}+x_{42}+x_{43}+x_{44}=10$$

$$x_{11}+x_{21}+x_{31}+x_{41}=10$$
$$x_{12}+x_{22}+x_{32}+x_{42}=10$$
$$x_{13}+x_{23}+x_{33}+x_{43}=10$$
$$x_{14}+x_{24}+x_{34}+x_{44}=10$$

$$x_{11}+x_{12}+x_{21}+x_{22}=10$$
$$x_{13}+x_{14}+x_{23}+x_{24}=10$$
$$x_{31}+x_{32}+x_{41}+x_{42}=10$$
$$x_{33}+x_{34}+x_{43}+x_{44}=10$$

Application - Sudoku Solving

2	4	x_{13}	x_{14}	$x_{11}+x_{12}+x_{13}+x_{14}=10$
x_{21}	1	x_{23}	2	$x_{21}+x_{22}+x_{23}+x_{24}=10$
1	x_{32}	x_{33}	4	$x_{31}+x_{32}+x_{33}+x_{34}=10$
x_{41}	x_{42}	1	3	$x_{41}+x_{42}+x_{43}+x_{44}=10$

$$x_{11}+x_{21}+x_{31}+x_{41}=10 \qquad x_{11}+x_{12}+x_{21}+x_{22}=10$$
$$x_{12}+x_{22}+x_{32}+x_{42}=10 \qquad x_{13}+x_{14}+x_{23}+x_{24}=10$$
$$x_{13}+x_{23}+x_{33}+x_{43}=10 \qquad x_{31}+x_{32}+x_{41}+x_{42}=10$$
$$x_{14}+x_{24}+x_{34}+x_{44}=10 \qquad x_{33}+x_{34}+x_{43}+x_{44}=10$$

$$x_{11}=2,\ x_{12}=4,\ x_{22}=1,\ x_{24}=2,\ x_{31}=1,\ x_{34}=4,\ x_{43}=1$$
$$x_{44}=3$$

Application - Sudoku Solving

```
i2 : gens gb ideal(x_11-2,x_12-4,x_22-1,x_24-2,x_34-4,x_43-1,x_44-3,x_31-1,x_11+x_12+x_21+x
_22-10,x_13+x_14+x_23+x_24-10,x_31+x_32+x_41+x_42-10,x_33+x_34+x_43+x_44-10,x_11+x_12+x_13+
x_14-10,x_21+x_22+x_23+x_24-10,x_31+x_32+x_33+x_34-10,x_41+x_42+x_43+x_44-10,x_11+x_21+x_31
+x_41-10,x_12+x_22+x_32+x_42-10,x_13+x_23+x_33+x_43-10,x_14+x_24+x_34+x_44-10)

o2 = | x_44-3 x_43-1 x_42-2 x_41-4 x_34-4 x_33-2 x_32-3 x_31-1 x_24-2 x_23-4 x_22-1 x_21-3
     ------------------------------------------------------------------------------------------
     x_14-1 x_13-3 x_12-4 x_11-2 |
```

Application – Sudoku Solving

```
i2 : gens gb ideal(x_11-2,x_12-4,x_22-1,x_24-2,x_34-4,x_43-1,x_44-3,x_31-1,x_11+x_12+x_21+x
_22-10,x_13+x_14+x_23+x_24-10,x_31+x_32+x_41+x_42-10,x_33+x_34+x_43+x_44-10,x_11+x_12+x_13+
x_14-10,x_21+x_22+x_23+x_24-10,x_31+x_32+x_33+x_34-10,x_41+x_42+x_43+x_44-10,x_11+x_21+x_31
+x_41-10,x_12+x_22+x_32+x_42-10,x_13+x_23+x_33+x_43-10,x_14+x_24+x_34+x_44-10)

o2 = | x_44-3 x_43-1 x_42-2 x_41-4 x_34-4 x_33-2 x_32-3 x_31-1 x_24-2 x_23-4 x_22-1 x_21-3
     ------------------------------------------------------------------------------------------
     x_14-1 x_13-3 x_12-4 x_11-2 |
```

2	4	3	1
3	1	4	2
1	3	2	4
4	2	1	3

Outline

Introduction

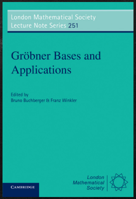
Gröbner Bases

Applications of Gröbner Bases

Conclusion

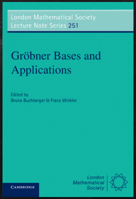
Summary

- Gröbner Bases are a very powerful notion with applications in (a) solving Diophantine equations (b) automated geometry theorem proving (c) signal and image processing (d) robotics (e) Sudoku puzzles (f) extrapolating "missing links" in palaeontology, TO BE CONTINUED



Summary

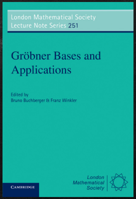
- Gröbner Bases are a very powerful notion with applications in (a) solving Diophantine equations (b) automated geometry theorem proving (c) signal and image processing (d) robotics (e) Sudoku puzzles (f) extrapolating “missing links” in palaeontology, TO BE CONTINUED



- Gives critical insight into ANY system of polynomials

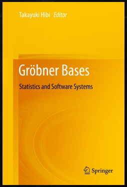
Summary

- Gröbner Bases are a very powerful notion with applications in (a) solving Diophantine equations (b) automated geometry theorem proving (c) signal and image processing (d) robotics (e) Sudoku puzzles (f) extrapolating "missing links" in palaeontology, TO BE CONTINUED



- Gives critical insight into ANY system of polynomials

- Lots of software to compute Gröbner bases - Macaulay2, Sage, Singular, CoCoA, etc.



References

T. Becker and V. Weispfenning.
Gröbner bases: a computational approach to commutative algebra, volume 141.
Springer Science & Business Media, 2012.

B. Buchberger and F. Winkler.
Gröbner bases and applications, volume 17.
Cambridge University Press Cambridge, 1998.

D. A. Cox, J. Little, and D. O'Shea.
Ideals, varieties, and algorithms.
Undergraduate Texts in Mathematics. Springer, Cham, fourth edition, 2015.
ISBN 978-3-319-16720-6; 978-3-319-16721-3.
doi: 10.1007/978-3-319-16721-3.

T. Hibi.
Gröbner bases: Statistics and software systems.
Springer Science & Business Media, 2014.