Gröbner Bases Native to 'pseudo'-Hodge Algebras, with an Application to Bideterminants

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- The answer to the question above required cleverness

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- UNDERSTATEMENT: Gröbner bases are very useful

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- CCT - classifying computational problems according to their resource usage, and understanding the classification itself
- Called 'quantitative epistemology', CCT has revealed profound insights into the mathematical universe, and has given P vs NP
- The algebraic version of P vs NP is VP vs VNP
- Determinantal complexity (dc) of a polynomial P: minimum $r \in \mathbb{N}$ such that there is an $r \times r$ matrix M of affine linear forms satisfying $\det(M) = P$

Computational Complexity Theory (CCT)

- example:

$$y + 2x + xz + 2xy - x^2z = \det \begin{pmatrix} x & y & 0 \\ -1 & z + y + 2 & x \\ 1 & z & 1 \end{pmatrix}$$

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- Conjecture: $\mathrm{dc}(\mathrm{perm}_n) = n^{\omega(1)}$

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- Gröbner bases are well-suited to both of the above: they give theoretical insight as well as are the key tool in effective methods

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- Obtaining a Gröbner basis can be tremendously expensive - EXPSPACE-complete (Mayr and Meyer [1982])
- Hard to get even simple cases to finish, e.g. 3×3 determinant orbit closure, tensor rank of 3×3 multiplication
- Gröbner bases tend to *obscure symmetry*!

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- There are Gröbner basis methods to obtain Algebraic de Rham Cohomology (Oaku and Takayama [1999, 2001])
- One can obtain lower bounds on decision tree complexity by obtaining Betti numbers of semi-algebraic sets
- Requires computing Gröbner bases of D-ideals (ideals in the Weyl algebra):

$$W_n := \mathbb{C}\left[\left\{X_i, \frac{\partial}{\partial X_i}\right\}_{i\in[n]}\right] \Big/ \left\langle \left\{\frac{\partial}{\partial X_i}\cdot X_i - X_i \cdot \frac{\partial}{\partial X_i} - 1\right\}_{i\in[n]} \right\rangle$$

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Question

Develop a Gröbner basis theory which takes advantage if variety corresponding to ideal has large symmetry group, or is 'determinantal'

Hodge Algebra (Alg. with Straightening Law)

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- If A is an ASL, the product of two standard monomials can be straightened into a linear combinaton of 'smaller' standard monomials
- ASLs arise as coordinate rings of algebraic varieties, e.g. Grassmanians, determinantal varieties, flag varieties, Schubert varieties

Bideterminants (products of minors)

Denoting

$$(\, r_1, \dots, r_k \mid c_1, \dots, c_k \,) := \det\left(\begin{bmatrix} x_{r_1,c_1} & \cdots & x_{r_1,c_k} \\ \vdots & \ddots & \vdots \\ x_{r_k,c_1} & \cdots & x_{r_k,c_k} \end{bmatrix}\right),$$

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The combinatorial datum below (bitableau)

$$\left[\begin{array}{|c|c|ccc|c|}\hline r_1^{(1)} & r_1^{(2)} & & & & r_1^{(p)} \\ \hline \vdots & \vdots & & & & \vdots \\ \hline \vdots & \vdots & & \cdots & & r_{\lambda_p}^{(p)} \\ \hline \vdots & r_{\lambda_2}^{(2)} & & & & \\ \hline r_{\lambda_1}^{(1)} & & & & & \\ \hline \end{array}\;\middle|\;\begin{array}{|c|c|ccc|c|}\hline c_1^{(1)} & c_1^{(2)} & & & & c_1^{(p)} \\ \hline \vdots & \vdots & & & & \vdots \\ \hline \vdots & \vdots & & \cdots & & c_{\lambda_p}^{(p)} \\ \hline \vdots & c_{\lambda_2}^{(2)} & & & & \\ \hline c_{\lambda_1}^{(1)} & & & & & \\ \hline \end{array}\right],$$

defines the following product of minors (bideterminant)

$$(R \mid C) := \prod_{i=1}^{p} \left(r_1^{(i)}, \ldots, r_{\lambda_i}^{(i)} \,\middle|\, c_1^{(i)}, \ldots, c_{\lambda_i}^{(i)} \right).$$

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- Thus Standard bideterminants form an $\mathbb{F}$-linear basis of $\mathbb{F}\left[\{X_{i,j}\}_{i\in[m],j\in[n]}\right]$

Bideterminants give Hodge Algebra

- Define
$$A = \mathbb{F}[X_{1,1}, X_{1,2}, X_{2,1}, X_{2,2}, Y] \Big/ \langle X_{1,2}X_{2,1} - X_{1,1}X_{2,2} + Y \rangle$$
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- (A, Σ) - the algebra of bideterminants - is an example of a Hodge algebra
- As seen before, $A \cong \mathbb{F}[X_{1,1}, X_{1,2}, X_{2,1}, X_{2,2}]$

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- This generalizes
  - poly ring with one variable for each minor of $n \times m$ matrix
  - quotient by relations between minors
  - gives ASL structure to the co-ordinate ring of $n \times m$ matrices
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  - standard monomials correspond to standard bitableaux
- Advantage - smaller expressions for 'determinant-like' polynomials; bideterminants are reflect symmetries coming from the action (representation theory) of GL_n

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Question

Can we build a theory of Gröbner bases 'native' to p-ASLs, i.e. Gröbner theory without referencing the ideal J?

Challenges

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- Product of standard monomials not necessarily standard, might require straightening
  - How do you define term order?
  - How would you define division of monomials?
  - What plays the role of monomial ideals?

Term Order & Division

- A p-ASL term order on a p-ASL A is a total order $\prec$ on standard monomials in A such that
  - $1 \preceq m$
  - If $a \prec b$ and $c \preceq d$, and $ac, bd \neq 0$, then
$$\mathrm{LM}(ac) \prec \mathrm{LM}(bd)$$

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  - If $a \prec b$ and $c \preceq d$, and $ac, bd \neq 0$, then
  $$\mathrm{LM}(ac) \prec \mathrm{LM}(bd)$$
- When does standard monomial m divide m':
  - ordinary division in the polynomial ring, or
  - m divides m' if there exists standard monomial f such that
  $$\mathrm{LM}(mf) = m'$$

Auxilliary Algebra of Leading Terms

- Given p-ASL A, algebra of leading terms w.r.t. A is another p-ASL A_{lt} on the same variables, and the same standard monomials such that for standard monomials m, m'

$$\pi_{\mathrm{lt}}(m) \cdot \pi_{\mathrm{lt}}(m') = \overbrace{0}^{\text{no straightening}} \quad \text{or} \quad \underbrace{\pi_{\mathrm{lt}}(\mathrm{LT}(mm'))}_{\text{leading term of straightening}}$$

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  where $\pi_{\mathrm{lt}} : A \to A_{\mathrm{lt}}$ is the identity

Proposition

Every p-ASL A admits two algebras of leading terms – A_{gen} where the product is never 0, and, A_{disc} where product is 0 unless $\mathfrak{m}\mathfrak{m}'$ is also a standard monomial.

Definition of p-ASL Gröbner Basis

- Given p-ASL A, algebra of leading terms A_{lt}, and an ideal $I \subseteq A$, then $G \subseteq A$ is a p-ASL Gröbner basis if:
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  - $\langle\{\pi_{\mathrm{lt}}(LM(g)) : g \in G\}\rangle = \langle\{\pi_{\mathrm{lt}}(LM(f)) : f \in I\}\rangle$ (standard monomial ideals in A_{lt})

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Existence	✓	✓
Reduced	✓	✓
Universal	✓	✓
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Corollary (Grochow-N, 2025)

The algebra of bideterminants has a p-ASL term order, thus we have a Gröbner basis theory (called bd-Gröbner bases).

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For any r, the set of minors of size $\geqslant r$ is a universal bd-Gröbner basis for its ideal.

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Takeaway

1. *Given all our machinery, the proof is one-line*
2. *In the ordinary case, universal Gröbner basis are known only for maximal minors and minors of size 2*

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- Get bd-Gröbner bases of annihilating D-ideals
- Compute Weyl closure, b-functions, etc. using bd-Gröbner bases in the Weyl algebra
- See if we can develop a bipermanent Gröbner basis theory (codimension of singular locus of permanent hypersurface is unknown!)

References

E. W. Mayr and A. R. Meyer. The complexity of the word problems for commutative semigroups and polynomial ideals. *Advances in mathematics*, 46(3):305–329, 1982.

T. Oaku and N. Takayama. An algorithm for de Rham cohomology groups of the complement of an affine variety via D-module computation. *J. Pure Appl. Algebra*, 139(1-3):201–233, 1999. doi: 10.1016/S0022-4049(99)00012-2.

T. Oaku and N. Takayama. Computing de Rham cohomology groups. In *Proceedings of the 33rd Symposium on Ring Theory and Representation Theory (Shimane, 2000)*, pages 19–22. Tokyo Univ. Agric. Technol., Tokyo, 2001.