Betti Numbers of Random Hypersurface Arrangements

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Joint work with Saugata Basu, Antonio Lerario

Outline

Introduction

Topology of Arrangement of Random Polynomials

References

Complexity of Arrangements

- Arrangement - finite collection of geometric objects

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- Analysis of arrangements of algebraic sets, i.e. $\bigcup_{i=1}^{s} Z(P_i)$ - important research area with applications (Agarwal-Sharir 2000)

- Knowledge of the Betti numbers of arrangements, has been used for understanding "combinatorial complexity" (Basu 2002)

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  - $b_1(X) = \#$one-dimensional or *circular* holes
  - $b_2(X) = \#$two-dimensional *voids* or *cavities*, etc.

Betti Numbers - Examples

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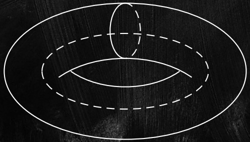	1	2	1	0

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- They offer a measure of complexity – e.g. height of algebraic computation tree for membership in semialgebraic set is lower bounded in terms of the Betti numbers (Yao 1997)
- In applications in incidence geometry, computational geometry, etc., especially for polynomial partitioning, bounds on Betti numbers of semi-algebraic sets are very important

Previous work on Arrangements

- Sum of Betti nos. (Oleinik-Petrovski (1949), Thom (1965), Milnor (1964)) - $P_1, \ldots, P_s \in \mathbb{R}[X_1, \ldots, X_n]$, max degree d

$$\sum_{j \geqslant 0} b_j \left(\bigcup_{i=1}^{s} Z(P_i) \right) = O(s^n d^n)$$

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Question

What are the expected Betti numbers of an arrangement of random polynomials?

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$$P(X_0, \ldots, X_n) = \sum_{\substack{\alpha=(\alpha_0,\ldots,\alpha_n) \\ \sum_{i=0}^n \alpha_i = d}} \xi_\alpha x_0^{\alpha_0} \ldots x_n^{\alpha_n},$$

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where $\xi_\alpha \sim \mathcal{N}\left(0, \frac{d!}{\alpha_0! \ldots \alpha_n!}\right)$ are independent
- Orthogonally-invariance: for any $L \in O(n+1, \mathbb{R})$,
$$P(X) \equiv_{\text{dist.}} P(LX)$$
- No points or directions are preferred in projective space

Expected Topology of Random Arrangements

Theorem (Basu-Lerario-N 2019)

Let $P_1, \ldots, P_s \in \mathbb{R}[X_0, \ldots, X_n]$ *be homogeneous Kostlan forms, each of degree at most* d. *Let* $\Gamma = \bigcup_{i=1}^{s} Z(P_i)$. *Then*

$$\mathbb{E}\left[b_0(\mathbb{RP}^n \setminus \Gamma)\right] = 2s^n d^{n/2} + O\left(s^{n-1} d^{(n-1)/2}\right).$$

Also, for $0 < i \leqslant n-1$

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Interpretation

Worst-case bound on b_0 *is* $\binom{s}{n} O(d^n)$, *while expectation is equal to* $2s^n d^{n/2}$.

Mayer-Vietoris Spectral Sequence

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Theorem (see for e.g. Basu [2003a])

There exists a first quadrant cohomological spectral sequence $(E_r, \delta_r)_{r \in Z}$, *where*

$$E_r = \bigoplus_{p,q \in \mathbb{Z}} E_r^{p,q}, \qquad \textit{and} \qquad E_0^{p,q} = \bigoplus_{\alpha_0 < \ldots < \alpha_p} C^q(A_{\alpha_0,\ldots,\alpha_p}),$$

with morphisms

$$\delta_r : E_r^{p,q} \to E_r^{p+r,q-r+1},$$

where

$$E_{r+1} \cong H_{\delta_r}(E_r).$$

This spectral sequence collapses at E_n *and converges to the cohomology of the union.*

Random Spectral Sequence

Proposition

Let $A_1, \dots, A_s$ be random simplicial complexes. Consider the same definitions as before. For every $r \geqslant 0$, define $e_r^{a,b} := \mathbb{E}\left[\text{rank } E_r^{a,b}\right]$. We have

$$e_{r+1}^{p,q} \leqslant e_r^{p,q},$$

and, if $E_r^{p+r,q-r+1} = 0$,

$$e_{r+1}^{p,q} \geqslant e_r^{p,q} - e_r^{p-r,q+r-1}.$$

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Proof.

$$E_{r+1}^{p,q} \cong \mathrm{Ker}(\delta_r : E_r^{p,q} \to E_r^{p+r,q-r+1}) \Big/ \mathrm{Img}(\delta_r : E_r^{p-r,q+r-1} \to E_r^{p,q}).$$

■

Proof of arrangements theorem

We need

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First ($k \geqslant 2$),

$$\sum_{k\geqslant 2}^{n} e_\infty^{n-k,k-1} \leqslant \sum_{k\geqslant 2}^{n} e_1^{n-k,k-1}$$
$$\leqslant s^{n-1} O(d^{(n-1)/2}),$$

because of Gayet-Welschinger (2015) (for any $p < n-1$):

$$e_1^{p,q} \leqslant \binom{s}{p+1} O(d^{(n-p-1)/2}).$$

Proof of arrangements theorem – contd...

Now it remains to give precise bounds on $e_\infty^{n-1,0}$:

$$e_\infty^{n-1,0} = e_n^{n-1,0} \leqslant e_1^{n-1,0} = 2s^n d^{n/2},$$

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$$\begin{aligned} e_\infty^{n-1,0} = e_n^{n-1,0} &\geqslant e_{n-1}^{n-1,0} - e_{n-1}^{0,n-2} \\ &\geqslant e_{n-1}^{n-1,0} - e_1^{0,n-2} \\ &\geqslant e_{n-2}^{n-1,0} - e_{n-2}^{1,n-3} - e_1^{0,n-2} \\ &\vdots \\ &\geqslant e_1^{n-1,0} - \left(\sum_{i=0}^{n-2} e_1^{i,n-2-i}\right) \\ &\geqslant 2s^n d^{n/2} - O\left(s^{n-1} d^{(n-1)/2}\right). \end{aligned}$$

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Just put everything together now. ■

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- $S \subseteq \mathbb{R}^n$ defined by $\{P_i \geqslant 0\}_{i \in [s]}$, $\deg(P_i) \leqslant 2$ (Barvinok 1997)

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Question

What is the expected Betti number of a union of random quadrics?

b_0 of Quadrics' Arrangement

Theorem (Basu-Lerario-N 2019)

Let $P_1, \ldots, P_s \in \mathbb{R}[X_0, \ldots, X_n]$ *be homogeneous Kostlan quadrics. Let* $\Gamma_i \subset \mathbb{RP}^n$ *be the zero set of* P_i*, and define* $\Gamma = \bigcup_{i=1}^{s} \Gamma_i$*. Then*

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$$\lim_{s\to\infty} \frac{\mathbb{E}\left[b_0(\Gamma)\right]}{s} = 0.$$

Interpretation

Our general theorem suggests $\mathbb{E}\left[b_0(\Gamma)\right] = O(s)$. For quadrics, we prove $\mathbb{E}\left[b_0(\Gamma)\right] = o(s)$.

Quadrics Arrangement - Proof

- Let $\mathrm{Sym}(n+1,\mathbb{R})$ be the vector space of $(n+1)\times(n+1)$ real symmetric matrices; we have

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- $\mathbb{RP}^N = \mathbb{P}(\mathrm{Sym}(n+1, \mathbb{R}))$ - projectivization of the space of symmetric matrices (here $N = \binom{n+2}{2} - 1$)
- Turns out sampling a Kostlan quadric is equivalent to sampling uniformly at random from S^N

Characterization of 'Quadrics' Intersection

Theorem (Calabi 1964)

For $n \geqslant 1$ *let* $q_1, q_2 \in \mathbb{R}[x_0, \dots, x_n]_{(2)}$ *and denote by* $\Gamma_1, \Gamma_2 \subset \mathbb{RP}^n$ *their (possibly empty) zero sets. Let* $\mathcal{P}_n \subseteq S^N$ *denote the set of positive quadratic forms. Let* $\ell \subset S^N$ *be the projective line* $\ell = \{[\lambda_1 q_1 + \lambda_2 q_2]\}_{\lambda_i \in \mathbb{RP}^1}$ *(a pencil of quadrics). Then:*

$$\Gamma_1 \cap \Gamma_2 \neq \emptyset \iff \ell \cap \mathcal{P}_n = \emptyset.$$

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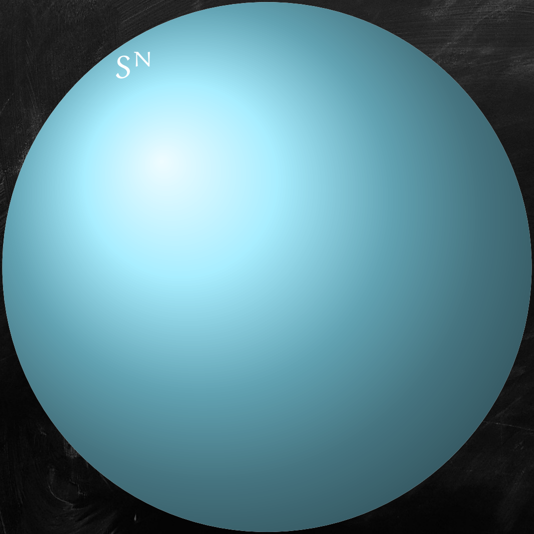
Our sampling process is equivalent to a random graph:

- *Sample s points uniformly at random from S^N*
- *Join points iff the great circle joining points does not pass through $\mathcal{P}_n$*

Illustration of 'Obstacle' Random Graph



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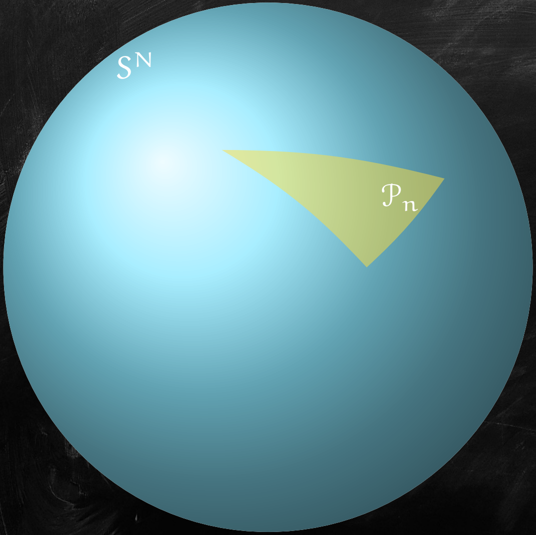

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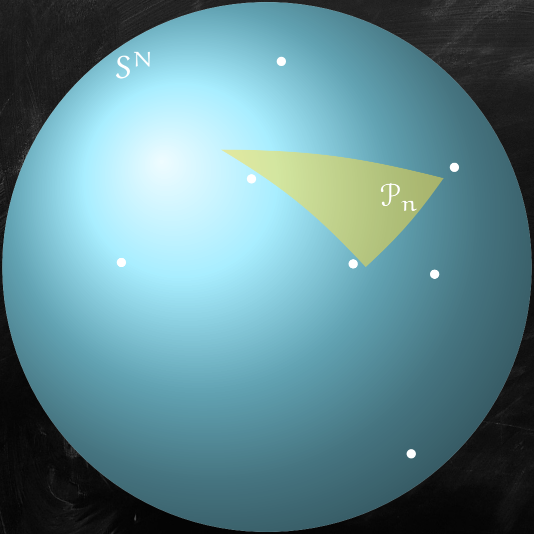

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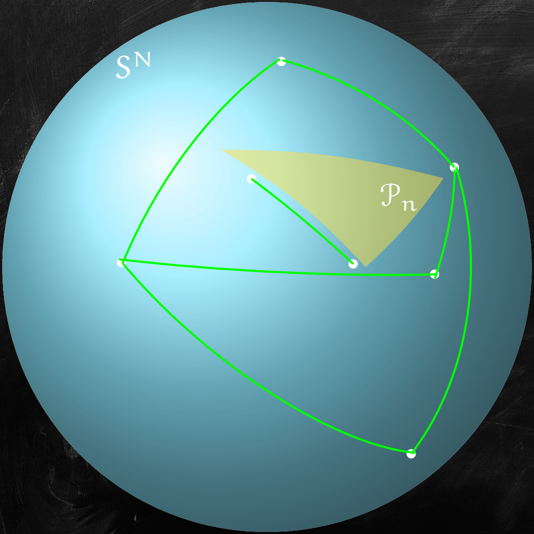

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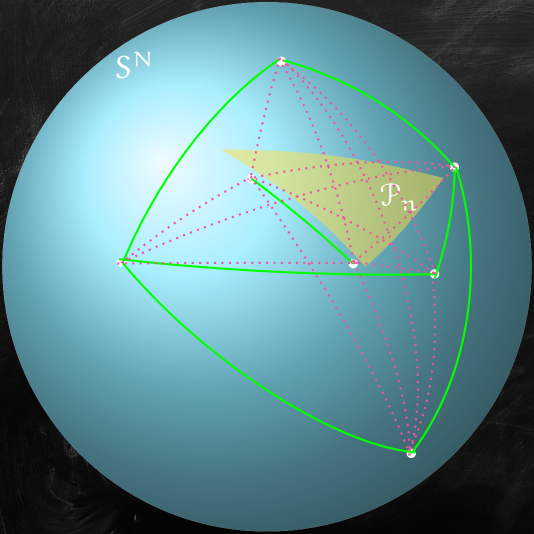

Obstacle Random Graph – Properties

- Good cone: for $q \in S^N$
$$g_q(\mathcal{P}_n) = \left\{x \in S^N \mid \ell(q, x) \cap \mathcal{P}_n = \emptyset\right\}.$$

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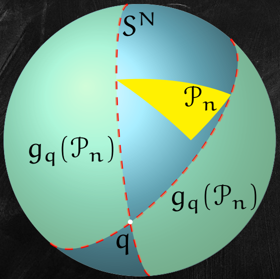

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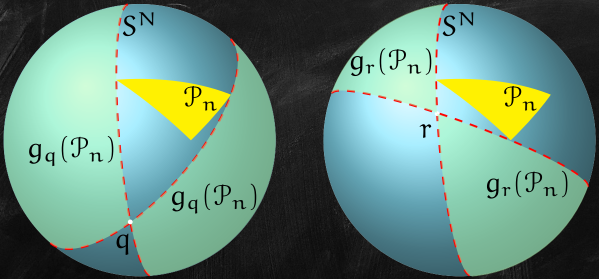

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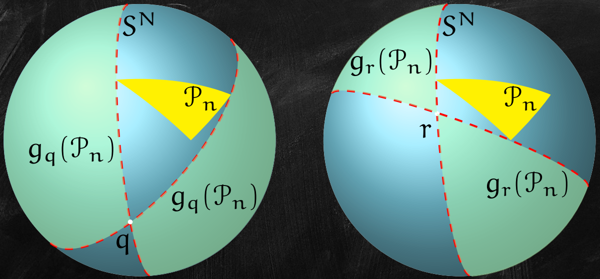

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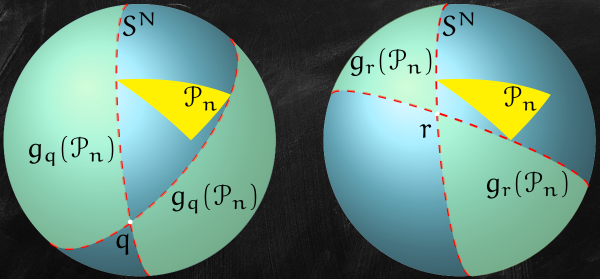

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- Probability random variables are not independent

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Question

What is the average number of connected components in the above random graph?

Average Connected Components

Theorem (Basu-Lerario-N 2019)

The expected number of connected component of $\mathcal{G}(N, \mathcal{P}_n, s)$ *satisfies:*

$$\lim_{s \to \infty} \frac{\mathbb{E}\left[b_0(\mathcal{G}(N, \mathcal{P}_n, s))\right]}{s} \leqslant \frac{\mathrm{vol}\left(\mathcal{P}_n\right)}{\mathrm{vol}\left(\mathbb{RP}^N\right)}.$$

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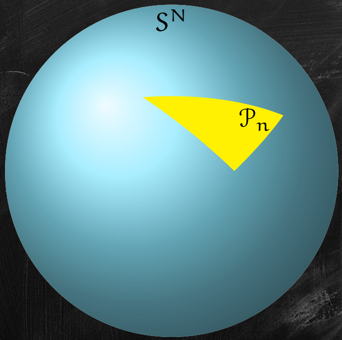
Interpretation

Considering $\frac{\mathrm{vol}(\mathcal{P}_n)}{\mathrm{vol}(S^N)}$ *to be fixed, we have that the expected number of connected components is* $o(s)$.

Average Connected Components – Proof



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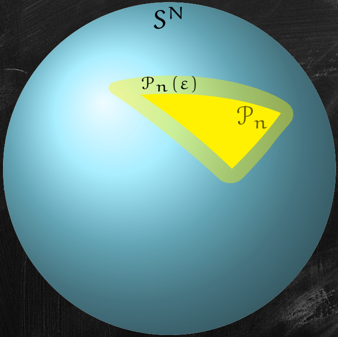

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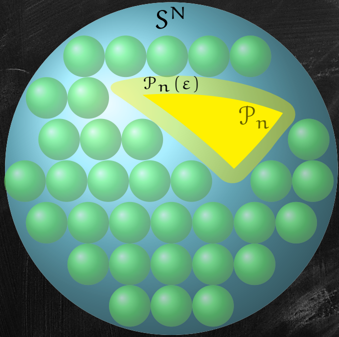

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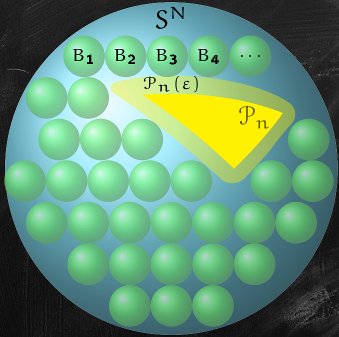

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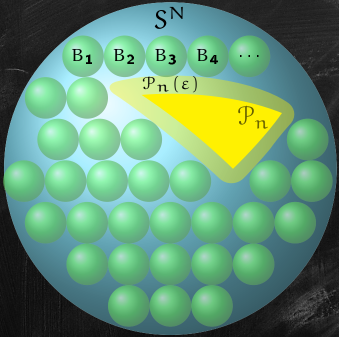

- For any $B_i \subseteq \mathcal{P}_n(\varepsilon)^c$, there exists $G_i \subseteq \mathcal{P}_n(\varepsilon)^c$,

$$\mu(G_i) > 0 \qquad \text{and} \qquad \forall p \in G_i,\ g_p(\mathcal{P}_n) \supseteq B_i.$$

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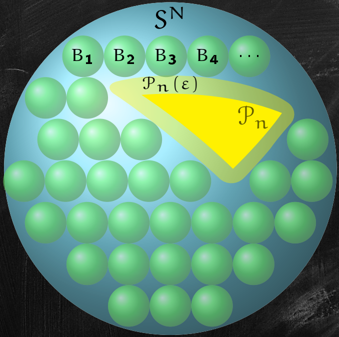

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- Using coupon-collector type argument, bound number of samples required to collect all B_i. ■

Future Work

- Expected Betti numbers of sign conditions on tuples of polynomials

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- Show strong bounds on the average number of connected components, at least for certain restricted types of obstacles
- Other questions about this random graph model
- Ramsey-theoretic results about Γ

Ramsey-Theoretic Result

Corollary (of Theorem on $b_0(\Gamma)$ for quadrics)

Let Γ be the graph of s quadrics. Then, for any $\varepsilon > 0$,

$$\lim_{s\to\infty} \mathbb{P}\left[\Gamma^c \text{ contains a clique of size } \varepsilon s\right] = 0.$$

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Theorem (Alon et al. [2005])

For any semi-algebraic graph $G = (V, E)$, there exists a constant $\delta > 0$, such that one of the following is true:

1. *There exists a clique of size $|V|^\delta$ in G.*
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Interpretation

Large cliques are impossible in Γ^c.

Outline

Introduction

Topology of Arrangement of Random Polynomials

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