Zeros of Polynomials on Definable Hypersurfaces -- pathologies exist, but they are rare

Abhiram Natarajan

Joint work with Saugata Basu and Antonio Lerario

Algebraic Techniques in Combinatorial Geometry

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- One technique, called polynomial partitioning, has helped solve problems in combinatorial and computational geometry (e.g. incidences, cycle elimination)
- Polynomial partitioning is a divide-and-conquer technique - simple and works well in any dimension

Basic Algebro-Geometric Definitions

- Algebraic Set: The locus of common zeros of $\{P_1, \ldots, P_s\}$, $P_i \in \mathbb{R}[X_1, \ldots, X_n]$, i.e.

$$Z(P_1, \ldots, P_s) := \{x \in \mathbb{R}^n \mid P_1(x) = \ldots = P_s(x) = 0\}$$

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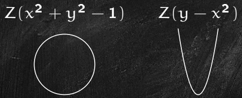

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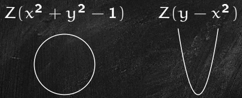

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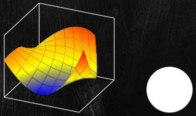
  - 2-dimensional 

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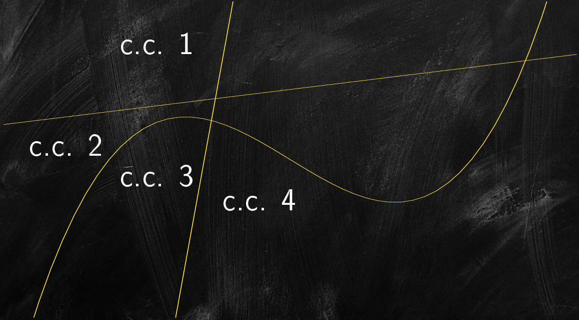

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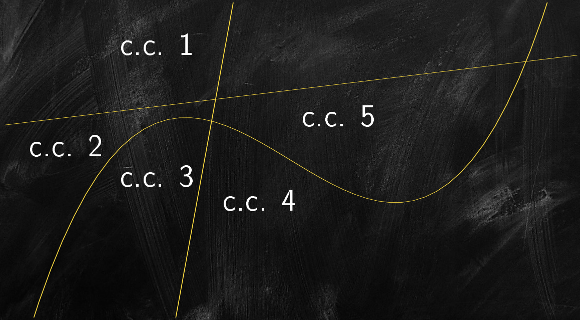

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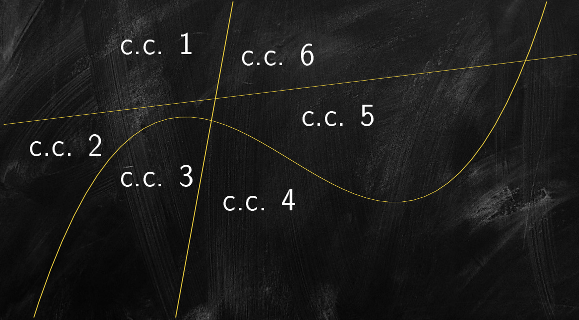

Polynomial Partitioning

Theorem (Guth and Katz [2015], Guth [2015])

Let Γ *be a finite set of* k*-dimensional algebraic sets in* $\mathbb{R}^n$*. For any* $D \geqslant 1$*, there is a polynomial* P *of degree* D*, each cell induced by* $Z(P)$ *intersects at most* $\sim \frac{|\Gamma|}{D^{n-k}}$ *algebraic sets of* Γ*.*

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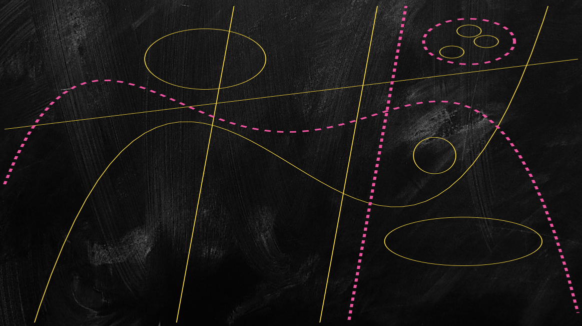

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See survey by Sharir [2017] for wide range of applications

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☆ Elements of $\mathcal{S}_1$ are precisely finite unions of points and intervals

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Question

Can we generalize polynomial partitioning to the o-minimal setting?

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Question

Given a definable hypersurface γ, *and a degree* D *polynomial* $P \in \mathbb{R}[X_1, \ldots, X_n]$, *how many cells induced by* P *does* γ *enter?*

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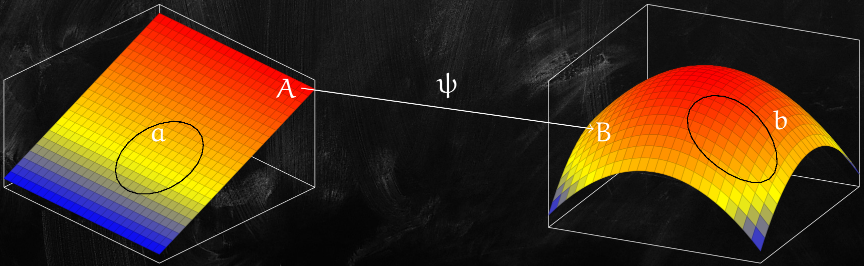

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  - $b_2(X) =$ #two-dimensional *voids* or *cavities*, etc.

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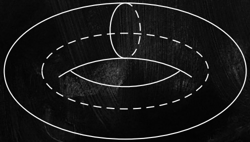	1	2	1	0

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- They offer a measure of complexity - Height of algebraic computation tree for membership in semialgebraic set is lower bounded in terms of the Betti numbers (Yao [1997], Gabrielov and Vorobjov [2017])

I donut like this joke!

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Definable Hypersurfaces ∩ Varieties

Informal Theorem

You can make the Betti numbers of the intersection of a definable hypersurface and an algebraic set arbitrarily large.

[1] To appear in Quarterly Journal of Mathematics, 2019

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Theorem (Existence of Pathologies - Basu-Lerario-N (2018)[1])

Let $\{Z_d\}_{d\in\mathbb{N}}$ *be a sequence of smooth, compact hypersurfaces in* $\mathbb{R}^{n-1}$. *There exists a regular, compact, semianalytic hypersurface* $\Gamma \subset \mathbb{RP}^n$, *a disk* $D \subset \Gamma$, *and a sequence* $\{p_m\}_{m\in\mathbb{N}}$ *of homogeneous polynomials with* $\deg(p_m) = d_m$ *such that the intersection* $Z(p_m) \cap \Gamma$ *is stable and:*

$$(D, Z(p_m) \cap D) \sim (\mathbb{R}^{n-1}, Z_{d_m}) \quad \textit{for all } m \in \mathbb{N}.$$

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Proof of Existence of Pathologies – Tools

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Theorem (Thom's Isotopy Lemma)

Suppose for a disk $D \subset \mathbb{R}^{n-1}$, *a regular hypersurface* $Z(f)$, $(D, D \cap Z(f)) \sim (\mathbb{R}^{n-1}, Z)$. *There exists* $\delta = \delta(f) > 0$ *such that for any regular function* $h : \overline{D} \to \mathbb{R}$ *with* $\|h\|_{C^1} \leqslant \delta$,

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Theorem (Seifert [1936])

Given a regular, compact hypersurface $Z \subset D \subset \mathbb{R}^{n-1}$*, there exists a polynomial* $q : \mathbb{R}^{n-1} \to \mathbb{R}$ *such that* $Z(q)$ *is regular and*

$$(D, Z(q)) \sim (\mathbb{R}^{n-1}, Z).$$

Proof of Existence of Pathologies - Key Ideas

- Recall we need definable Γ and polynomials $P_1, P_2, \ldots$ s.t.
  $\Gamma \cap Z(P_1) \approx Z_{d_1}$, $\Gamma \cap Z(P_2) \approx Z_{d_2}$, ...

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  - By Thom's Isotopy lemma, if $\| \sum_{j \geqslant k+2} Q_j \|_{C^1}$ is bounded, $Z_{d_k} \approx Z(Q_{k+1}) \approx \Gamma \cap Z(P_k)$ ■

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Question

How 'common' is the pathological case? What does 'common' even mean?

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where $\xi_\alpha \sim \mathcal{N}\left(0, \frac{d!}{\alpha_0! \ldots \alpha_n!}\right)$ are independent
- Distribution is orthogonally-invariant, i.e for any matrix L satisfying $L^T L = L L^T = I$,
$$P(X) \equiv_{\text{dist.}} P(LX)$$

Average Topology of Definable Hypersurfaces on Algebraic Sets

Theorem (Measure of Pathologies - Basu-Lerario-N (2018)[2])

Let $\Gamma \subset \mathbb{RP}^n$ *be a regular, compact semi-analytic hypersurface, and let* p *be a random Kostlan polynomial of degree* D*. Then there exists a constant* c_Γ *such that for every* $0 \leqslant k \leqslant n-2$*, for every* $t > 0$

$$\mathbb{E}\left[b_k(\Gamma \cap Z(p))\right] = c_\Gamma D^{n-1/2}.$$

Proof Technique: Morse Theory + Kac-Rice Formula

[2] To appear in Quarterly Journal of Mathematics, 2019

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Future Questions:

- Ambitiously, can we can prove that the measure of partitioning polynomials for a given Γ is large, then there exists a partitioning polynomial that is not pathological for any Γ?
- Instead of algebraic partitioning hypersurfaces, can we use definable partitioning hypersurfaces?

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