Partitioning Theorems for Sets of Semi-Pfaffian Sets, with Applications

Abhiram Natarajan

Collaborators: Prof. Martin Lotz (Univ. of Warwick), Prof. Nicolai Vorobjov (Univ. of Bath)

Real Algebraic Geometry

- Real Algebraic Set: The set of real zeros of polynomials $P_1, \ldots, P_s \in \mathbb{R}[X_1, \ldots, X_n]$

$$Z(P_1, \ldots, P_s) := \{x \in \mathbb{R}^n \mid P_1(x) = \ldots = P_s(x) = 0\}$$

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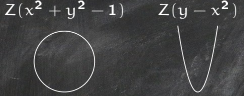

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$\{-(x^2 + y^2 - 1) \geqslant 0\}$

$\{y \geqslant x\} \wedge \{x \geqslant y\}$

$\{x^2 + y^2 \leqslant 2\} \wedge (\{y - x \geqslant 4\} \vee \neg\{x - y \leqslant 4\})$

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- Algebro-geometric techniques have been very effective
- Technique called polynomial partitioning has helped solve several open problems in incidence geometry and other areas

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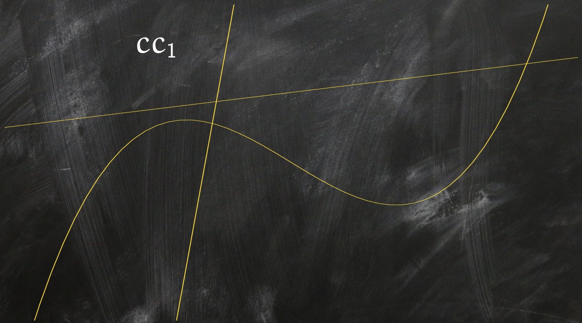

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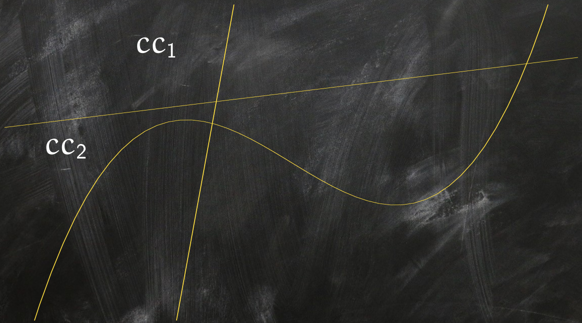

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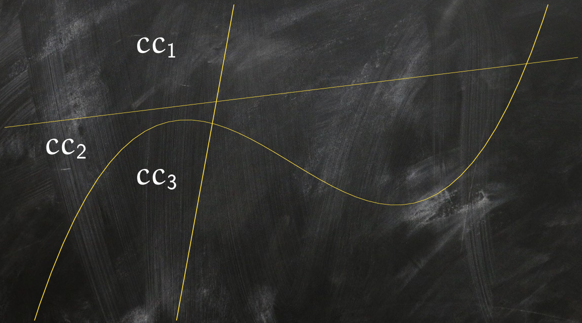

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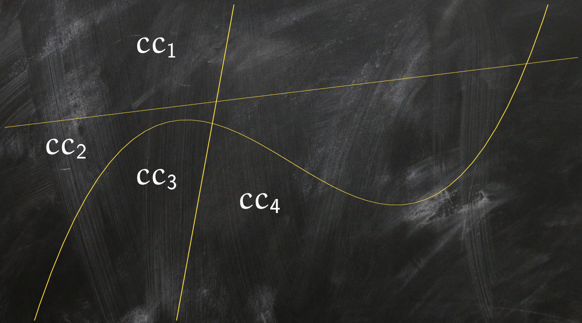

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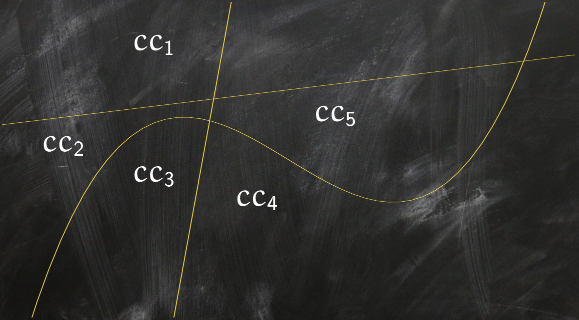

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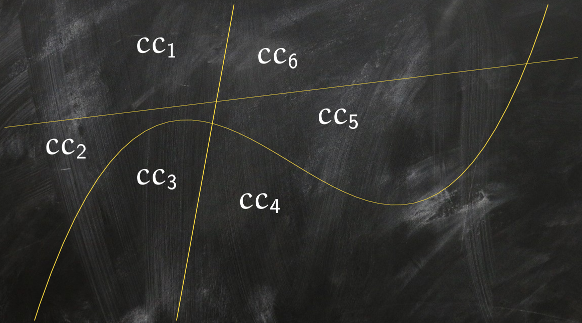

Polynomial Partitioning

Theorem (Guth and Katz [2015], Guth [2015])

Let Γ *be a set of* k*-dimensional semialgebraic sets in* $\mathbb{R}^n$*. For any* $D \geqslant 1$*, there is a polynomial* P *of degree* $\leqslant D$*, such that each cell induced by* P *intersects at most* $\sim \frac{|\Gamma|}{D^{n-k}}$ *elements of* Γ*.*

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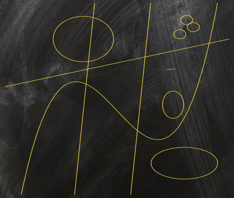

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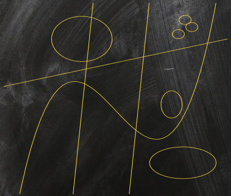

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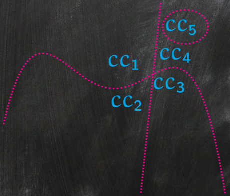

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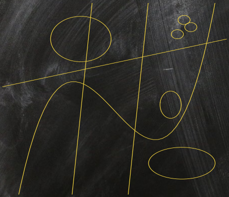

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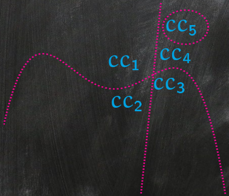

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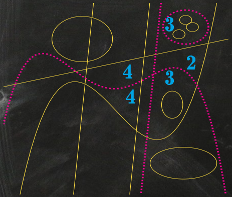

each cell intersects only few curves from Γ

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See survey by Kaplan et al. [2012] for a wide range of applications

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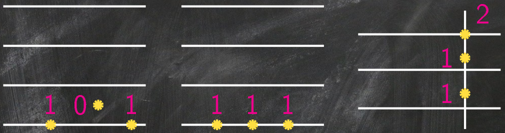
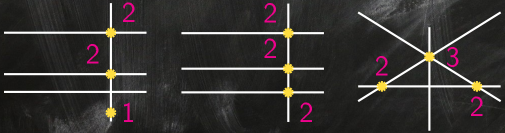


2 incidences 3 incidences 4 incidences



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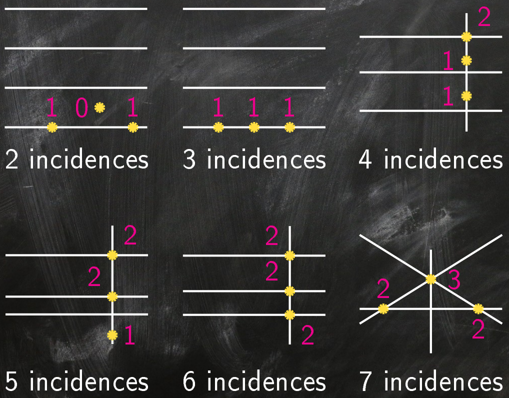

- m points and n lines in $\mathbb{R}^2$: $\mathcal{O}(m^{2/3}n^{2/3} + m + n)$ incidences

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Takeaway

Polynomial partitioning and basic arguments worked!

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- O-minimal geometry (geometry of definable sets) is an axiomatic generalization of real algebraic geometry

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Question

Can we generalize polynomial partitioning to the o-minimal setting? ... we make progress...

Pfaffian Functions

- Let $\mathcal{U} \subseteq \mathbb{R}^n$ be an open set. $\vec{f} = (f_1, \ldots, f_r)$, $f_i \in \mathcal{C}^\infty(\mathcal{U})$ is a Pfaffian chain if there exist polys. $P_{i,j} \in \mathbb{R}[X_1, \ldots, X_n, Y_1, \ldots, Y_i]$ verifying
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- $g : \mathbb{R}^n \to \mathbb{R}$ is a Pfaffian function w.r.t. $\vec{f}$ if there exists polynomial $Q_g \in \mathbb{R}[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ such that
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$$g(x) = Q_g(x, f_1(x), \dots, f_r(x))$$
- $\alpha := \max_{i,j} P_{i,j}$ is called chain-degree, $\beta := \deg(Q_g)$ is degree of g, r is order of g; (α, β, r) is format of g

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- $\vec{q} = \left(\frac{1}{x}, \ln(x)\right)$ is a Pfaffian chain on the domain $R \setminus \{0\}$; any $P \in \mathbb{R}\left[X, \frac{1}{X}, \ln(X)\right]$ is a Pfaffian function.

Pfaffian Functions – Examples

- A polynomial of degree D w.r.t. the empty chain; format is $(\alpha, D, 0)$ for any integer $\alpha > 0$.

- $\vec{q} = (q_1, \ldots, q_r)$, where $q_i(x) = e^{q_{i-1}(x)}$, and $q_0(x) = ax$, is a Pfaffian chain of order r and chain-degree r. Consequently, any $P \in \mathbb{R}[X, e^{aX}, e^{e^{aX}}, \ldots]$ is a Pfaffian function w.r.t. $\vec{q}$.

- $\vec{q} = (\frac{1}{x}, \ln(x))$ is a Pfaffian chain on the domain $R \setminus \{0\}$; any $P \in \mathbb{R}\left[X, \frac{1}{X}, \ln(X)\right]$ is a Pfaffian function.

- $\vec{q} = (\frac{1}{x}, x^m)$ for any $m \in \mathbb{R}$ is a Pfaffian chain; any $P \in \mathbb{R}\left[X, \frac{1}{X}, X^m\right]$ is a Pfaffian function w.r.t. $\vec{q}$.

Pfaffians

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- The Pfaffian structure, i.e., the smallest collection of sets containing all semi-Pfaffian sets and that is stable under all structure operations, is an o-minimal structure.

Main Theorem

Theorem (Partitioning Pfaffians [Lotz-N-Vorobjov, 2024])

Let Γ be a collection of semi-Pfaffian sets in $\mathbb{R}^n$ of dimension k, where each $\gamma \in \Gamma$ has order r.

1. *For any $D \geqslant 1$, there is $P \in \mathbb{R}[X_1, \ldots, X_n]$ of degree D, such that each cell induced by P intersects at most $\frac{|\Gamma|}{D^{n-k-r}}$ elements of Γ.*

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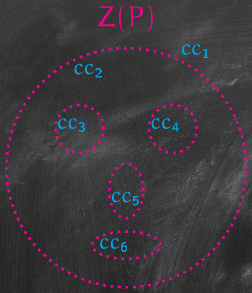
Takeaway

1. *Generalization of Polynomial Partitioning to Pfaffians*
2. *New technique of Pfaffian Partitioning*

Proof - Main Technical Step



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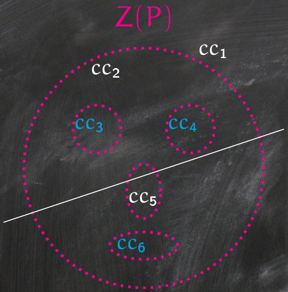

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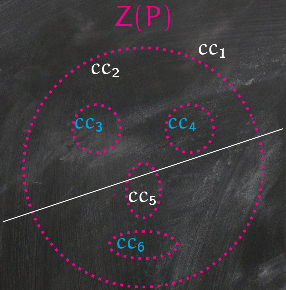

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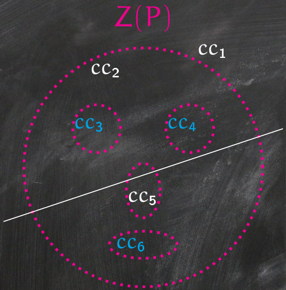

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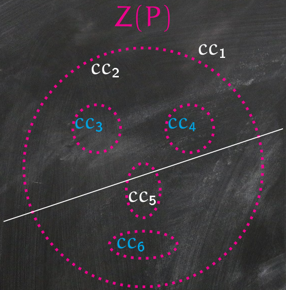

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- Poly. P of deg. D in n variables induces at most D^n cells
- We show for a k-dimensional semi-Pfaffian set γ of order r

  γ intersects at most D^{k+r} cells induced by P

Applications of Partitioning Pfaffians

- (Pfaffian Szemerédi-Trotter) m points and n Pfaffian curves of order r in $\mathbb{R}^2$: $\mathcal{O}(m^{\frac{2r+2}{2r+3}+\varepsilon}n^{\frac{r+2}{2r+3}}+m+n)$ incidences

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- We also count joints between Pfaffian curves
- More applications possible
- Our technique lends itself to generalizing to other o-minimal structures with caveat

Crucial Ingredient – Khovanskii's Theorem

Theorem ([Khovanskiĭ, 1991, §3.12, Corollary 5])

Let $\vec{q}$ be a Pfaffian chain of order r and chain-degree α, where the functions in the Pfaffian chain depend only on $\xi \leqslant n$ variables. Let $f_1, \ldots, f_n$ be Pfaffian functions on an open set $\mathcal{U} \subseteq \mathbb{R}^n$, where f_i is of degree β_i w.r.t. $\vec{q}$. The number of non-degenerate solutions of $\{x \in \mathcal{U} : f_1(x) = \ldots = f_n(x) = 0\}$ is bounded from above by

$$2^{\binom{r}{2}} \beta_1 \ldots \beta_n \left(\min\{\xi, r\}\alpha + \beta_1 + \ldots + \beta_n - n + 1\right)^r.$$

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Takeaway

Bézout type theorem holds for Pfaffian sets.

Definable Hypersurfaces ∩ Varieties

Theorem (Basu-Lerario-N 2018)

Let $\{Z_d\}_{d\in\mathbb{N}}$ *be a sequence of smooth, compact hypersurfaces in* $\mathbb{R}^{n-1}$*. There exists a regular, compact, semianalytic hypersurface* $\Gamma \subset \mathbb{RP}^n$*, a disk* $D \subset \Gamma$*, and a sequence* $\{p_m\}_{m\in\mathbb{N}}$ *of homogeneous polynomials with* $\deg(p_m) = d_m$ *such that the intersection* $Z(p_m) \cap \Gamma$ *is stable and:*

$$(D, Z(p_m) \cap D) \sim (\mathbb{R}^{n-1}, Z_{d_m}) \quad \textit{for all } m \in \mathbb{N}.$$

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Takeaway

You can make the Betti numbers of the intersection of a definable hypersurface and an algebraic set arbitrarily large.

References

S. Barone and S. Basu. Refined bounds on the number of connected components of sign conditions on a variety. *Discrete & Computational Geometry*, 47(3):577–597, 2012.

S. Basu, A. Lerario, and A. Natarajan. Zeroes of polynomials on definable hypersurfaces: pathologies exist, but they are rare. *Quarterly Journal of Mathematics (to appear)*, 2018.

L. Guth. Polynomial partitioning for a set of varieties. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 159, pages 459–469. Cambridge University Press, 2015.

L. Guth and N. H. Katz. On the erdős distinct distances problem in the plane. *Annals of Mathematics*, pages 155–190, 2015.

H. Kaplan, J. Matoušek, and M. Sharir. Simple proofs of classical theorems in discrete geometry via the guth–katz polynomial partitioning technique. *Discrete & Computational Geometry*, 48: 499–517, 2012.

A. G. Khovanskiĭ. *Fewnomials*, volume 88. American Mathematical Soc., 1991.

M. Lotz, A. Natarajan, and N. Vorobjov. Partitioning theorems for sets of semi-pfaffian sets, with applications, 2024. URL `https://arxiv.org/abs/2412.02961`.

J. Milnor. On the betti numbers of real varieties. *Proceedings of the American Mathematical Society*, 15(2):275–280, 1964.

O. Oleinik and I. Petrovsky. On the topology of real algebraic hypersurfaces. *Izv. Acad. Nauk SSSR*, 13:389–402, 1949.

R. Thom. Sur l'homologie des variétés algébriques réelles. *Differential and combinatorial topology*, pages 255–265, 1965.